Inhomogeneous universe models

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Abstract
A short introduction to differential geometry and general relativity is presented in order to provide the reader with enough knowledge to follow the rest of the thesis. The Lemaître-Tolman-Bondi metric is introduced and the field equations are found using Cartan's formalism. It is shown that an LTB universe with radiation and cosmological constant as the only content must be homogeneous. Even more generally, it is shown that for an LTB universe with $p(r,t) = \omega(t)\rho(r,t)$ and $-1 \neq \omega \neq 0$, the metric is reduced to the Friedmann-Robertson-Walker metric. An LTB-model with radiation, dust, cosmological constant and an interaction term modeled as anisotropic pressure is also studied and a differential equation governing the time evolution is derived. The differential equation is solved in the special case $\Lambda = 0$ and the solutions are analysed and plotted. The only factorizable solutions of the LTB metric with the source $T^{\mu\nu} = \rho(t,r)diag[1, \omega(t)+2\alpha(t), \omega(t)-\alpha(t), \omega(t)-\alpha(t)]$, $0 \neq \omega \neq -\frac{1}{2}\alpha$ and a cosmological constant is derived. The spatial geometry of these solutions is studied and two-dimensional hyper-surfaces are plotted in 3D-plots.

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Chapter 1

Introduction

Observational data on high red-shift supernovae from the end of the last millennium indicate a positive cosmological constant and that the universe is expanding at an accelerating rate [6, 7]. The Nobel prize in physics for 2011 was awarded to Saul Perlmutter, Brian P. Schmidt and Adam G. Riess for this discovery.

The luminosity of type 1a supernovae vary very little and hence from their apparent magnitude it is possible to calculate the distances to these supernovae. Comparing their redshift-distance relationship with that in a universe with unaccelerated expansion velocity and assuming that the universe is homogeneous, one found that the expansion must accelerate.

There is however a different possibility. The distance-redshift relation may be partly due to spatial inhomogeneity. This means that the expansion is slower far away than near us, instead of slower far back in time.

Observations are to a high degree isotropic, i.e. the observations are the same in all directions away from us. This means that an eventual inhomogeneity must be spherically symmetric with us close to the centre of symmetry.

This motivates the study of the Lemaître-Tolman-Bondi metric, which is a general spherically symmetric metric. There has been several publications on this subject, but it seemed that a purely radiation dominated model had yet not been thoroughly investigated. As the universe is believed to have undergone a period of evolution when the amount of dust was negligible in comparison to the amount of radiation, this may be of interest.

The investigation of a radiation dominated, spherically symmetric model is the starting point for this thesis.

Outline

Before making any independent analysis, the theoretical framework which is needed is reviewed in chapter 2. Differential geometry, which is the mathematical framework of general relativity, is introduced and eventually applied in the theory itself.

Chapter 3 introduces the LTB metric and the Einstein equations for this metric are found using Cartan's formalism.

In chapter 4 the field equations are solved for a universe with radiation and a cosmological constant. It is shown that the only possible solution for this type of source is a homogeneous universe.

Chapter 5 adds dust to the energy-momentum tensor as well as an interaction term entering the equations as anisotropic pressure. The analysis is very similar to a previous work which did not include a cosmological constant. The solution of the equations for $\Lambda = 0$ but without assuming zero curvature (which was done in the previous work) is presented and the different solutions are analysed and plotted.

Chapter 6 considers a general perfect fluid as well as anisotropic pressure proportional to the energy density and a cosmological constant. Every factorizable, inhomogeneous, non-dust solution for this type of source is found. The spatial sections are analysed and two dimensional hyper-surfaces are plotted in 3D plots.
It is shown that for zero anisotropic pressure, every position-independent ratio between pressure and energy density except 0 and -1 reduces to the homogeneous metric. The conclusion of chapter 4 is a special case of this.

Finally, the conclusions are presented in chapter 7 and prospects for further research are mentioned.

Chapter 2

Introduction to General Relativity

This chapter aims to give the reader enough knowledge of the general theory of relativity to comprehend the rest of this thesis. A more thorough introduction can be found in e.g. the introductory book [4], which was my own introduction to general relativity.

2.1 Mathematical preliminaries

2.1.1 Manifolds

General relativity stands out from many physical theories in the way that it does not only concern the evolution of physical objects in a rigid, euclidean space. It describes the dynamics of space as well.

Manifolds are exactly the mathematical objects that can represent various space-times which seem euclidean locally, but may have other geometrical properties as a whole.

For a more thorough introduction to differential geometry, see e.g. [9]. A short introduction can also be found in [4]

A **topological space** a is set X and a set $\mathcal{O}$ of subsets of S with the following properties:

- $X, \emptyset \in \mathcal{O}$
- $\mathcal{O}$ is closed under arbitrary unions
- $\mathcal{O}$ is closed under finite intersections

The subsets contained in $\mathcal{O}$ are said to be the open subsets of X and the complement of an open set is said to be closed.

An **n-dimensional manifold** or **n-manifold**, M, is a topological space which is locally homeomorphic to $\mathbb{R}^n$, that is, for any $p \in M$ one can find an open subset U of M, containing p, such that there is a continuous bijection between U and $\mathbb{R}^n$.

The property that the manifold can be covered with open subsets suggests a natural way to keep track of the points in the manifold. If U is an open subset of M and x is a homeomorphism (continuous bijection) $U \rightarrow \mathbb{R}^n$, then a point p in U can be associated with the coordinate $x(p)$. A collection of such coordinate maps, or **charts**, whose domains cover the whole of M is called an **atlas** of M. The pair (x, U) will be used to denote a chart with domain U and $x(p) \in \mathbb{R}^n$ will be written $x^\mu(p)e_\mu$ where e_μ are basis vectors and $x^\mu(p)$ are components.

Continuous functions from manifolds to other spaces are well defined on ordinary manifolds as defined above. A function $f : M \rightarrow N$ is continuous if the inverse image of every open set in N is open in M. However, differentiability is not well defined.

Intuitively, a function $f : M \rightarrow \mathbb{R}^m$ should be differentiable at p if, for a chart x with a domain including p, $f \circ x^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at $x(p)$. This idea needs to be refined, as if y is another chart, then $f \circ y^{-1}$ does not need to be differentiable even though $f \circ x^{-1}$ is. However, if $y \circ x^{-1}$ is differentiable, then $f \circ y^{-1}$ is differentiable at $y(p)$ if $f \circ x^{-1}$ is differentiable at $x(p)$ (whenever p is in the domain of x and in the domain of y). If we demand this for every pair of charts in our atlas, then differentiability will be well defined.

Having added this structure to the manifold, one might as well extend the atlas to a maximal one. That is, for any atlas one can include every chart that obeys the requirement above. It can be shown that this gives a unique **maximal atlas**

A **differentiable manifold** is a manifold together with a maximal atlas, $\mathcal{A}$, where $x \circ y^{-1}$ is differentiable $\forall\, x, y \in \mathcal{A}$. Similarly, a **C^k-, C$^\infty$- or C$^\omega$ manifold** is a manifold with a maximal atlas under the requirement that $y \circ x^{-1}$ is either k times differentiable, smooth or analytic respectively.

It is not unusual to use the phrase "differentiable manifold" to mean C^∞ manifold. For the purposes of general relativity, the structure of a differentiable manifold is necessary. From now on, when discussing manifolds, it will be assumed that they are at least sufficiently many times differentiable.

2.1.2 The tangent bundle

Having established a structure for differentiating functions on the manifold, it would be profitable to define vectors and vector fields.

Let I denote the unit interval, $[0,1]$. Consider a point p on a manifold, M, and a chart (x, U) covering p. Let $c \ : \ I \ \rightarrow \ U, \ c(\frac{1}{2}) = p$ be a curve through p with $x \circ c$ differentiable. $x \circ c : I \ \rightarrow \mathbb{R}^n$ will similarly denote a curve in $\mathbb{R}^n$ and $\frac{d}{d\lambda}(x(c(\lambda))_{|\lambda=\frac{1}{2}}$ will be a vector in $\mathbb{R}^n$.

One can form an equivalence relation $\approx$ defined by

$$\frac{d}{d\lambda}(x(c_1(\lambda))_{|\lambda=\frac{1}{2}} = \frac{d}{d\lambda}(x(c_2(\lambda))_{|\lambda=\frac{1}{2}} \Leftrightarrow c_1 \approx c_2. \tag{2.1}$$

This divides all such curves into equivalence classes. However, these classes are are restricted to curves in the domain of the chart (x, U). Intending to define coordinate independent structures, one would like to define an equivalence relation on the set of all curves through p taking values in the domain of any chart.

It turns out that this is straight forward as if (x, U) and (y, V) are two charts covering p, then $U \cap V$ is a neighbourhood of p. Suppose $c_1 : I \rightarrow U, \ c_2 : I \rightarrow V$, then the definition (2.1) will still be valid. It remains however to show that the definition is independent of chart, that is

$$\frac{d}{d\lambda}(x(c_1(\lambda))_{|\lambda=\frac{1}{2}} = \frac{d}{d\lambda}(x(c_2(\lambda))_{|\lambda=\frac{1}{2}} \Leftrightarrow \frac{d}{d\lambda}(y(c_1(\lambda))_{|\lambda=\frac{1}{2}} = \frac{d}{d\lambda}(y(c_2(\lambda))_{|\lambda=\frac{1}{2}} \tag{2.2}$$

This follows as the manifold is differentiable, as this gives

$$\frac{d}{d\lambda}(y(c(\lambda))_{|\lambda=\frac{1}{2}} = D(y \circ x^{-1})\frac{d}{d\lambda}(x(c(\lambda))_{|\lambda=\frac{1}{2}}, \tag{2.3}$$

giving the implication to the right, and similarly

$$\frac{d}{d\lambda}(x(c(\lambda))_{|\lambda=\frac{1}{2}} = D(x \circ y^{-1})\frac{d}{d\lambda}(y(c(\lambda))_{|\lambda=\frac{1}{2}}, \tag{2.4}$$

for the converse.

From now, the equivalence relation $\approx$ will be defined in the coordinate independent way.

The **Tangent plane of M at p**, T_p is defined as $\{c \ : \ I \ \rightarrow \ U, \ c(\frac{1}{2}) = p\}/\approx$ with a vector space structure defined in the natural way, associating each element of T_p with the derivative evaluated at $\frac{1}{2}$ of one of it elements composed with x. Hence, the elements of T_p are called vectors at p.

The **Tangent bundle of M**, **TM** is defined as $\cup_{p\in M} T_p$.
A **vector field**, or a **section** of the tangent bundle, is a subset of TM containing exactly one element from each tangent plane. This can also be regarded as a function $V : M \to TM$ with $V(p) \in T_p$.

Even though the structure is coordinate independent, it is of course useful to assign coordinates to the manifold. Also, one needs something to represent the elements of the tangent bundle. There is a very natural way to use a chosen coordinate system in order to represent the vectors and vector fields.

Assume (x,U) is a chart defined at p, $v \in T_p$ and c is a curve in the equivalence class v. One may represent $\boldsymbol{v} \in T_p$ by $\tilde{\boldsymbol{v}} = \frac{d}{d\lambda}(x(c(\lambda)))_{|\lambda=\frac{1}{2}} \in \mathbb{R}^n$.

As it makes notation easier, $\boldsymbol{v}$ will be identified with $\tilde{\boldsymbol{v}}$ and thus also T_p is identified with $\mathbb{R}^n$. This means that

$$\boldsymbol{v} = \tilde{\boldsymbol{v}} = \frac{d}{d\lambda}(x^\mu(c(\lambda)))_{\lambda=\frac{1}{2}}\boldsymbol{e}_\mu \equiv v^\mu \boldsymbol{e}_\mu, \tag{2.5}$$

where $\boldsymbol{e}_\mu$ are basis vectors of $\mathbb{R}^n$. Also, **Einstein's summation convention** was introduced. Whenever a term contains two identical indices, these indices are summed over, e.g.

$$\sum_\mu x^\mu x_\mu \equiv x^\mu x_\mu. \tag{2.6}$$

If x' is another chart at p, then one should also have

$$\boldsymbol{v} = \frac{d}{d\lambda}(x'^\mu(c(\lambda)))_{\lambda=\frac{1}{2}}\boldsymbol{e}'_\mu \equiv v'^\mu \boldsymbol{e}'_\mu, \tag{2.7}$$

where the $\boldsymbol{e}'_\mu$ are the coordinate basis vectors of the range of x'.
This can also be written

$$v = \frac{d}{d\lambda}[x'^\mu(x^{-1}(x^\nu(c(\lambda))\boldsymbol{e}_\nu))]_{\lambda=\frac{1}{2}}\boldsymbol{e}'_\mu \equiv v'^\mu \boldsymbol{e}'_\mu. \tag{2.8}$$

Introducing the short-hand notation, $\frac{\partial x'^\mu}{\partial x'^\nu} \equiv \frac{\partial(x'^\mu \circ x^{-1})}{\partial x^\nu}$ and using the chain rule gives

$$\frac{d}{d\lambda}[x'^\mu(x^{-1}(x^\nu(c(\lambda))\boldsymbol{e}_\nu))]_{\lambda=\frac{1}{2}}\boldsymbol{e}'_\mu = \frac{\partial x'^\mu}{\partial x^\nu}\frac{d}{d\lambda}[x^\nu(c(\lambda))]_{\lambda=\frac{1}{2}}\boldsymbol{e}'_\mu = \frac{\partial x'^\mu}{\partial x^\nu}v^\nu \boldsymbol{e}'_\mu = v'^\mu \boldsymbol{e}'_\mu. \tag{2.9}$$

And the transformation rule

$$v'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}v^\nu \tag{2.10}$$

follows.

As $v^\mu \boldsymbol{e}_\mu$ and $v'^\mu \boldsymbol{e}'_\mu$ both have been identified with the same element of T_p, it follows that

$$v^\mu \boldsymbol{e}_\mu = v'^\nu \boldsymbol{e}'_\nu = v^\mu \frac{\partial x'^\nu}{\partial x^\mu} \boldsymbol{e}'_\nu \tag{2.11}$$

This gives the transformation rule for the basis vectors. It is customary to prime the indices instead of the components/basis vectors, when referring to a vector in different bases. This convention will be obeyed in the following. The rules for components and basis vectors are summarized in (2.12)

$$\boxed{v^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} v^\nu, \quad \boldsymbol{e}_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}} \boldsymbol{e}_\nu} \tag{2.12}$$

The vector components are said to transform *contravariantly*, while the basis vectors transforms *covariantly*. The latter transforms in the same way as differential operators, that is

$$\frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\nu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\nu}. \tag{2.13}$$

This allows for identifying basis vectors with differential operators.

$$\boldsymbol{e}_\mu \equiv \frac{\partial}{\partial x^\mu} \tag{2.14}$$

2.1.3 One-forms

If V is a vector space, **the dual space** V^* of V is defined as the set of all linear functions $\lambda\ : V \to \mathbb{R}$. An element of V^* is called a **one-form**.

V^* is also a vector space and if $\alpha, \beta \in V^*$, one defines addition and scalar multiplication in the natural way

$$(\boldsymbol{\alpha} + \boldsymbol{\beta})(\mathbf{v}) = \boldsymbol{\alpha}(\mathbf{v}) + \boldsymbol{\beta}(\mathbf{v}) \tag{2.15}$$

and

$$(a\boldsymbol{\alpha})(\mathbf{v}) = \boldsymbol{\alpha}(\mathbf{v}) \cdot a. \tag{2.16}$$

Given a set of basis vectors $\{\boldsymbol{e}_\mu\}$ for V, one defines a basis $\boldsymbol{\omega}_\mu$ of V^* by

$$\boldsymbol{\omega}^\mu(\boldsymbol{e}_\nu) = \delta^\mu_\nu. \tag{2.17}$$

A one-form can be written

$$\boldsymbol{\alpha} = \alpha_\mu \boldsymbol{\omega}^\mu \tag{2.18}$$

and

$$\boldsymbol{\alpha}(\boldsymbol{v}) = \alpha_\mu \boldsymbol{\omega}^\mu (v^\nu \boldsymbol{e}_\nu) = \alpha_\mu v^\nu \delta^\mu_\nu = \alpha_\mu v^\mu. \tag{2.19}$$

One also regards vectors in V as linear functions from V^* to $\mathbb{R}$ defined by

$$\boldsymbol{v}(\boldsymbol{\alpha}) \equiv \boldsymbol{\alpha}(\boldsymbol{v}) = v^\mu \alpha_\mu. \tag{2.20}$$

This interpretation is fundamental for the interpretation of tensors in the subsequent section.

The forms of interest will be the elements of the dual space of the tangent planes of M. The **dual bundle** of M is defined as

$$TM^* \equiv \cup_{p \in M} T_p^*. \tag{2.21}$$

The transformation properties of one-forms will also be important, for $\boldsymbol{\alpha}(\boldsymbol{v}) = \alpha_\mu v^\mu$ to be invariant, it is seen that the components of a one-form must transform covariantly. The basis one-forms must transform contravariantly.

As a general rule, entities with lower indices transform covariantly and entities with upper indices transform contravariantly.

2.1.4 Tensors

A **tensor** is a multilinear function, mapping vectors and forms to real numbers. Multilinearity means that a tensor is linear in each of its arguments, e.g.

$$T(\boldsymbol{a}+\boldsymbol{b}, \boldsymbol{c}+\boldsymbol{d}) = T(\boldsymbol{a}, \boldsymbol{c}) + T(\boldsymbol{a}, \boldsymbol{d}) + T(\boldsymbol{b}, \boldsymbol{c}) + T(\boldsymbol{b}, \boldsymbol{d}). \tag{2.22}$$

The discussion will be restricted to tensors with arguments from a single vector space and its dual. That is, tensors $T : V^n \times V^{*m} \to \mathbb{R}$.

The rank of a tensor mapping m forms and n vectors to $\mathbb{R}$ is denoted by $\{^m_n\}$ or sometimes by n+m. Such a tensor is said to be covariant if $m = 0$, contravariant if $n = 0$ and mixed if it is neither covariant nor contravariant.

The **tensor product** denoted by $\otimes$ is a binary function on the set of tensors. It is defined as

$$(T \otimes S)(u_1, u_2, ..., u_m, v_1, v_2, ..., v_n) = T(u_1, ...u_m)S(v_1, ..., v_n) \tag{2.23}$$

It is not hard to imagine that for any pair of non-negative integers (n,m) with $n + m > 0$, the set of tensors of rank $\left\{ {m \atop n} \right\}$ with a specific order of vector/form arguments forms a vector space.
This means that one can represent tensors as linear combinations of basis tensors.

If the tensors are required to take their form arguments before their vector arguments, the basis elements can be written on the form

$$\boldsymbol{e}_{\mu_1} \otimes ... \otimes \boldsymbol{e}_{\mu_m} \otimes \boldsymbol{\omega}^{\nu_1} \otimes ... \otimes \boldsymbol{\omega}^{\nu_n}, \tag{2.24}$$

and a tensor may be written as

$$T^{\mu_1...\mu_m}_{\nu_1...\nu_n} \boldsymbol{e}_{\mu_1} \otimes ... \otimes \boldsymbol{e}_{\mu_m} \otimes \boldsymbol{\omega}^{\nu_1} \otimes ... \otimes \boldsymbol{\omega}^{\nu_n}. \tag{2.25}$$

It is seen that the dimension of the vector space of tensors is $dim(V)^{m+n}$

A tensor is a coordinate independent object, so assigning the property "covariant" or "contravariant" to it is not very logical. The rule of thumb for remembering which tensors are covariant and which tensors are contravariant is that it is the tensor components that transform according to the property that is assigned to the tensor. The basis elements transform in the opposite way. The convention makes some sense as equations involving tensors often are written on component form.

A **scalar** can be thought of as a tensor of rank zero. If it is a scalar field, it is a function $M \to \mathbb{R}$. Otherwise, it may be a single value assigned to a point in M. It is explicitly coordinate independent as it has no basis vectors and only one "component". The value of a tensor evaluated at some element of $TM^n \times TM^{*m}$ is a scalar.

2.1.5 Forms

An **antisymmetric tensor**, A, is a tensor that changes sign under interchange of any two different arguments. This can of course not apply to a mixed tensor. As

$$A_{...\mu...\nu...} = A(..., \boldsymbol{e}_{\boldsymbol{\mu}}, ..., \boldsymbol{e}_{\boldsymbol{\nu}}, ...) = -A(..., \boldsymbol{e}_{\boldsymbol{\nu}}, ..., \boldsymbol{e}_{\boldsymbol{\mu}}, ...) = -A_{...\nu...\mu...}, \tag{2.26}$$

the components of such a tensor are antisymmetric as well.

A **p-form** is a covariant antisymmetric tensor of rank p.

It turns out that antisymmetric tensors also form vector spaces. However, defining a basis for such tensors is a bit more cumbersome than defining a basis for general tensors.

Let S(n) be the symmetric group, the group of permutations on n elements. Let A(n) be the alternating group, the subgroup of S(n) consisting of even permutations only. Also, define their group actions on the set $\{1, ..., n\}$.

The **exterior product** or **wedge product**, $\vee$ is a binary operation mapping a p-form and a q-form to a (p+q)-form. It is defined on basis forms as

$$\boldsymbol{\omega}^{\mu_1} \vee ... \vee \boldsymbol{\omega}^{\mu_p} = 2 \sum_{\sigma \in A(n)} \boldsymbol{\omega}^{\mu_{\sigma(1)}} \otimes ... \otimes \boldsymbol{\omega}^{\mu_{\sigma(p)}} - \sum_{\sigma \in S(n)} \boldsymbol{\omega}^{\mu_{\sigma(1)}} \otimes ... \otimes \boldsymbol{\omega}^{\mu_{\sigma(p)}}. \quad (2.27)$$

The exterior product is linear and associative and commutative/anti-commutative depending on the forms it operates on. If $\boldsymbol{\alpha}$ is a p-form and $\boldsymbol{\beta}$ is a q-form, then

$$\boldsymbol{\alpha} \vee \boldsymbol{\beta} = (-1)^{pq} \boldsymbol{\beta} \vee \boldsymbol{\alpha}. \quad (2.28)$$

It turns out that

$$\{\boldsymbol{\omega}^{\mu_1} \vee ... \vee \boldsymbol{\omega}^{\mu_p} | \mu_1 < ... < \mu_p\} \quad (2.29)$$

forms a basis for the p-forms. If the dimension of the vector space of one-forms is n, the dimension of the vector space of p-forms is $\binom{n}{p}$ for $p \leq n$ and 0 otherwise.

A p-form $\boldsymbol{\alpha}$ is expressed as a linear combination of basis elements as

$$\boldsymbol{\alpha} = \frac{1}{p!} \alpha_{\mu_1 ... \mu_p} \boldsymbol{\omega}^{\mu_1} \vee ... \vee \boldsymbol{\omega}^{\mu_p}, \quad (2.30)$$

where the factor $\frac{1}{p!}$ remedies that the Einstein summation convention does not exclusively sum over increasing indices.

2.1.6 Exterior differentiation of forms

In the preceding discussion on tensors and forms, it was not emphasized in particular that the discussed forms and tensors where part of a tensor field or a *form field*. This was done in order to isolate every vector space and assign a basis to them. However in general relativity, every tensor or form will be some tensor or form field evaluated at a point p in the spacetime manifold. The discussion from now will be concerning form fields rather than just forms. However, a form field on a manifold, M, can also be called a **form on M**. From here, what is meant by "form" will always be “form on M”.

A 0-form is a function $f : V^0 \to \mathbb{R}$ at every point in M, and can be interpreted as a function $f : M \to \mathbb{R}$. The **differential, df, of a 0-form, f**, is defined as

$$\boldsymbol{d}f = \frac{\partial f}{\partial x^\mu}\boldsymbol{\omega}^\mu. \tag{2.31}$$

Taking the differential of the coordinate functions, it is obtained that $\boldsymbol{d}x^\mu = \boldsymbol{\omega}^\mu$. The differential is a one-form with components $\frac{\partial f}{\partial x^\mu}$ as these components transform covariantly.

The **differential, d$\boldsymbol{\alpha}$, of a p-form, $\boldsymbol{\alpha}$**, is defined as

$$\boldsymbol{d\alpha} = \frac{1}{p!}\frac{\partial a_{\mu_1 \ldots \mu_p}}{\partial x^\mu}\boldsymbol{d}x^\mu \vee \boldsymbol{d}x^{\mu_1} \vee \ldots \vee \boldsymbol{d}x^{\mu_p} \tag{2.32}$$

It is easily seen that this is a (p+1)-form.

2.1.7 The metric tensor

The notion of *distance* is important in any physical theory which describes moving objects. On a manifold, distances are defined by means of a *metric tensor*.

The **metric tensor, $\boldsymbol{g}$** is a symmetric covariant tensor of rank 2.

$$\boldsymbol{g} = g_{\mu\nu}\boldsymbol{\omega}^\mu \otimes \boldsymbol{\omega}^\nu \tag{2.33}$$

It defines a *pseudo inner product* on the tangent planes of M,

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{g}(\boldsymbol{a}, \boldsymbol{b}) = g_{\mu\nu}a^\mu b^\nu, \tag{2.34}$$

in the sense that it fulfills every condition for an inner product except that it may be negative and a non-zero vector may give zero when dotted with itself.

This is the case in relativity when one considers metric tensors on a four dimensional manifold.

When expanded on diagonal form, the metric tensor will have one negative and three positive components. A manifold with such a metric tensor, or equivalently a metric tensor with one positive and three negative components is called a **Lorentzian manifold**.

The length, s, of a curve, $c : [a, b] \to M$ is defined as

$$s(c) = \int_a^b \sqrt{g(\boldsymbol{u}(\lambda), \boldsymbol{u}(\lambda))} d\lambda, \tag{2.35}$$

where $\boldsymbol{u}(\lambda)$ is the equivalence class of c at $c(\lambda)$, or in other words, the tangent vector of c. The expression under the square root of (2.35) will be referred to as the squared *length* of the vector $\boldsymbol{u}(\lambda)$. Vectors with negative, zero and positive squared length will be called time-like, light-like and space-like respectively.

The possibly negative inner product allows distances which are not real. However one usually calculates the length of curves which have constant sign on the inner product. When calculating the *proper time* of a particle moving at subluminal speed, the inner product will be negative exclusively and the **proper time, $\boldsymbol{\tau}$** is given by

$$\tau(c) = \int_a^b \sqrt{-g(\boldsymbol{u}(\lambda), \boldsymbol{u}(\lambda))} d\lambda. \tag{2.36}$$

The proper time is proportional to the time that lapses on a clock which travels along the curve. The proportionality constant can of course in theory be adjusted to 1.

The relation between distance and metric tensor makes it possible to write the metric tensor (if it is diagonal) in the expanded form

$$g = ds^2 = -d\tau^2 = g_{00}(dx^0)^2 + g_{11}(dx^1) + g_{22}(dx^2)^2 + g_{33}(dx^3)^2. \tag{2.37}$$

Here ds^2 etc. should be read as $\boldsymbol{ds} \otimes \boldsymbol{ds}$. This is the form which will be used when presenting a metric.

In addition to the covariant metric tensor, a similar contravariant symmetric tensor is defined,

$$\tilde{g} = g^{\mu\nu} e_\mu \otimes e_\nu. \tag{2.38}$$

Its components are defined by the relation

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu. \tag{2.39}$$

This means that the matrix representing the components of $\tilde{g}$ is the inverse of the matrix of the components of g. If the metric tensor is diagonal, this means that $g^{\mu\nu} = \frac{1}{g_{\mu\nu}}$.

These tensors will be used to create new tensors from old ones in processes of raising or lowering an index or by contraction of indices.

If T^μ_ν are the components of a tensor, one defines the tensors with components $T^{\mu\nu}$, $T_{\mu\nu}$ and T as

$$T^{\mu\nu} = T^\mu_\lambda g^{\lambda\nu},\ T_{\mu\nu} = T^\lambda_\nu g_{\lambda\mu},\ T = T^\mu_\mu. \tag{2.40}$$

2.1.8 Covariant differentiation of vectors

Differentiation of functions on a manifold was made possible when the differentiable structure was introduced. Exterior differentiation of forms has also been defined with the antisymmetry characteristic to p-forms. Differentiation of vector fields on the other hand is more tricky.

One wants the derivative to compare the vector field at points which are infinitesimally close. Given the direction of the derivative, it should return a vector, much in the same way as the derivative of a function is another function.
If $c(\lambda)$ is a curve on M and X is a vector field, the definition should be something like

$$\frac{d}{d\lambda}(X(c(\lambda))) \stackrel{?}{=} \lim_{\delta\lambda\to 0} \frac{X(c(\lambda+\delta\lambda)) - X(c(\lambda))}{\delta\lambda}, \tag{2.41}$$

however, this definition is not valid, as $X(c(\lambda+\delta\lambda)$ and $X(c(\lambda))$ belong to different tangent spaces. Merely identifying the basis vectors and comparing components is no good either as the basis vectors are dependent of the coordinate system and this makes the derivative coordinate dependent.

A useful shorthand notation is given by

$$\boldsymbol{e}_\nu(x^{\rho_1\cdots\rho_m}_{\sigma_1\ldots\sigma_n}) \equiv x^{\rho_1\cdots\rho_m}_{\sigma_1\ldots\sigma_n,\nu}. \tag{2.42}$$

This is generally not a tensor component. The $_{,\nu}$ index transforms like a tensor index, but the other indices don't. This is why it makes sense to differentiate functions but not vectors. For example, let A^μ be a tensor component.

$$A^{\mu'}_{,\nu} = \frac{\partial}{\partial x_\nu}(\frac{\partial x^{\mu'}}{\partial x^\nu}A^\mu) = \frac{\partial^2 x^{\mu'}}{\partial x^\nu \partial x^\nu}A^\mu + \frac{\partial x^{\mu'}}{\partial x^\mu}A^\mu_{,\nu} \tag{2.43}$$

The transformed non-tensor component contains an extra term, which distinguishes it from a tensor component.

This is resolved by introducing the concept of *parallel transport*. One will not delve rigorously into this concept, but merely state that the parallel transported vector $X_{||}(c(\lambda+\delta\lambda))$ from $c(\lambda+\delta\lambda)$ to $c(\lambda)$ is a vector in $T_{c(\lambda)}$ with components that coincide with the components of $X(c(\lambda + \delta\lambda))$ when evaluated in a locally Cartesian coordinate system.

The covariant derivative of X along the curve c at $c(\lambda)$ is interpreted as

$$\frac{d}{d\lambda}(X(c(\lambda))) = \lim_{\delta\lambda \to 0} \frac{X_{||}(c(\lambda + \delta\lambda)) - X(c(\lambda))}{\delta\lambda}. \tag{2.44}$$

If $\boldsymbol{y} = y^\mu \boldsymbol{e}^\mu$ is the equivalence class for c in $T_{c(\lambda)}$ it is written in component form as

$$\frac{d}{d\lambda}(X(c(\lambda))) = y^\nu X^\mu_{;\nu} \boldsymbol{e}_\mu. \tag{2.45}$$

In a locally Cartesian coordinate system this is just the ordinary derivative, the trick in the definition is that one has defined the derivative in a subset of the possible coordinate systems and these definitions are consistent with each other. The definition of the covariant derivative in other coordinate systems then follow from the tensor transformation properties.

One may think of the general covariant derivative of X as a mixed tensor of rank $\binom{1}{1}$. It is function that takes the vector y as an argument and returns the covariant derivative of X in the direction of y.

A semicolon will always be used to signify indices $_{;\nu}$ which emerge from covariant differentiation in the direction of $\boldsymbol{e}_\nu$.

The covariant derivative of $\boldsymbol{X}$ in direction $\boldsymbol{Y}$ will be denoted

$$Y^\nu X^\mu_{;\nu} \boldsymbol{e}_\mu \equiv \nabla_{\boldsymbol{Y}} \boldsymbol{X}. \tag{2.46}$$

If $\boldsymbol{Y}$ is a basis vector, the short hand notation

$$\nabla_\mu \equiv \nabla_{\boldsymbol{e}_\mu}. \tag{2.47}$$

will be used.

The covariant derivative of a vector component can be expressed as

$$\boxed{A^\mu_{;\nu} \equiv A^\mu_{,\nu} + A^\alpha \Gamma^\mu_{\alpha\nu}, \tag{2.48}}$$

where the $\Gamma^\mu_{\alpha\nu}$ are the **connection coefficients**. The connection coefficients are called **Christoffel symbols** when represented in a coordinate basis. The Christoffel symbols can be calculated directly from the metric according to the rule

$$\boxed{\Gamma^{\mu}_{\alpha\nu} = \frac{1}{2} g^{\mu\lambda}(g_{\lambda\alpha,\nu} + g_{\lambda\nu,\alpha} - g_{\alpha\nu,\lambda}).} \tag{2.49}$$

2.1.9 Covariant differentiation of tensors

The covariant derivative is generalised inductively to tensors of arbitrary rank. If f is a function (a tensor of rank 0), its covariant derivative is defined as

$$\nabla_{\boldsymbol{X}} f = \boldsymbol{X}(f). \tag{2.50}$$

It is just the ordinary derivative of a function.
For a one-form, the covariant derivative is defined as

$$(\nabla_{\boldsymbol{X}} \boldsymbol{\alpha})(\boldsymbol{A}) = \nabla_{\boldsymbol{X}}[\boldsymbol{\alpha}(\boldsymbol{A})] - \boldsymbol{\alpha}(\nabla_{\boldsymbol{X}} \boldsymbol{A}). \tag{2.51}$$

This is more or less according to the product rule for differentiation.
This is also the foundation for extending the definition to tensors of arbitrary rank. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are tensors, the covariant derivative of their tensor product is given by

$$\nabla_{\boldsymbol{X}}(\boldsymbol{A} \otimes \boldsymbol{B}) = (\nabla_{\boldsymbol{X}} \boldsymbol{A}) \otimes \boldsymbol{B} + \boldsymbol{A} \otimes (\nabla_{\boldsymbol{X}} \boldsymbol{B}) \tag{2.52}$$

The covariant derivative of the components of a contravariant tensor of rank 2 follows easily from this relation and the components of the covariant derivative of a vector (2.48).

$$\boxed{T^{\mu\nu}_{;\alpha} = T^{\mu\nu}_{,\alpha} + \Gamma^{\mu}_{\lambda\alpha} T^{\lambda\nu} + \Gamma^{\nu}_{\lambda\alpha} T^{\mu\lambda}} \tag{2.53}$$

This is particularly convenient for expressing conservation of energy and momentum in general relativity. Also, this shows that the divergence of the metric tensor is zero, $g^{\mu\nu}_{;\nu} = 0$.

2.1.10 Curvature and Cartan's formalism

Einsteins field equations relates the distribution of energy and pressure in the universe to the *curvature* of the universe.

The term "curvature" occurs in the names of several geometrical entities and is not defined as an entity on its own.

One type of curvature which is not too hard to understand is *Gaussian curvature*. Gaussian curvature is a property of two-dimensional surfaces which makes it suitable for comprehension. It can be defined in numerous ways, the definition to be presented here will be regarding the ratio between the radius and circumference of a circle.

Around any point p on the surface, one can define the circle of radius r centered at p as the set of points on the surface which are separated from p by a distance r. The distance between two points is understood as the smallest length of the curves connecting the two points. For sufficiently small r, the circle at p will be a closed curve.
Denote the length of this curve by $rC_p(r)$. It is well known that if the surface is a plane, the circumference is given by $rC_p(r) = 2\pi r$.

One definition [10] of **Gaussian curvature, $\boldsymbol{K(p)}$** is

$$K(p) = \lim_{r \to 0^+} 3\frac{2\pi r - rC_p(r)}{\pi r^3}. \tag{2.54}$$

On a smooth manifold, this can also be written as

$$K(p) = -\frac{3}{2\pi}C_p''(0). \tag{2.55}$$

A sphere of radius 1 has $C_p(r) = 2\pi\frac{sin(r)}{r}$ for $0 < r \leq \pi$ and $C_p(0) = 2\pi$. This gives a Gaussian curvature

$$\begin{aligned} &K(p) = \lim_{r\to 0^+} -3(\tfrac{-sin(r)}{r} - 2\tfrac{cos(r)}{r^2} + 2\tfrac{sin(r)}{r^3}) \\ &= 3 + 6\lim_{r\to 0^+} \tfrac{rcos(r)-sin(r)}{r^3} = 3 + 6\lim_{r\to 0^+} \tfrac{-rsin(r)+rcos(r)-rcos(r)}{3r^2} = 1. \end{aligned} \tag{2.56}$$

The Gaussian curvature is negative if the circumference of a circle tends to grow faster than $2\pi r$. as an example, consider the surface in $\mathbb{R}^3$ parametrised by $z = xy$. This is illustrated in figure 2.1.

Another entity which will be referred to as curvature appears explicitly in the metrics that will be studied in this thesis. The Friedmann-Robertson-Walker metric for homogeneous spaces can be written as

$$ds^2 = -dt^2 + a^2(t)(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2sin^2\theta d\phi^2) \tag{2.57}$$

The constant k will be referred to as the curvature of this space. It is somewhat similar to Gaussian curvature in the sense that the surface area of spheres increases faster with radius if the curvature is negative and slower with positive curvature. The surface area of spheres in flat space ($k = 0$) is $4\pi r^2$. In the Lemaître-Tolman-Bondi metric (LTB for short) for spherically symmetric universes, kr^2 is replaced by a function F which plays a similar role. It must be stressed, though that in the LTB metric, the origin is a special point and the position dependent F is defined relative

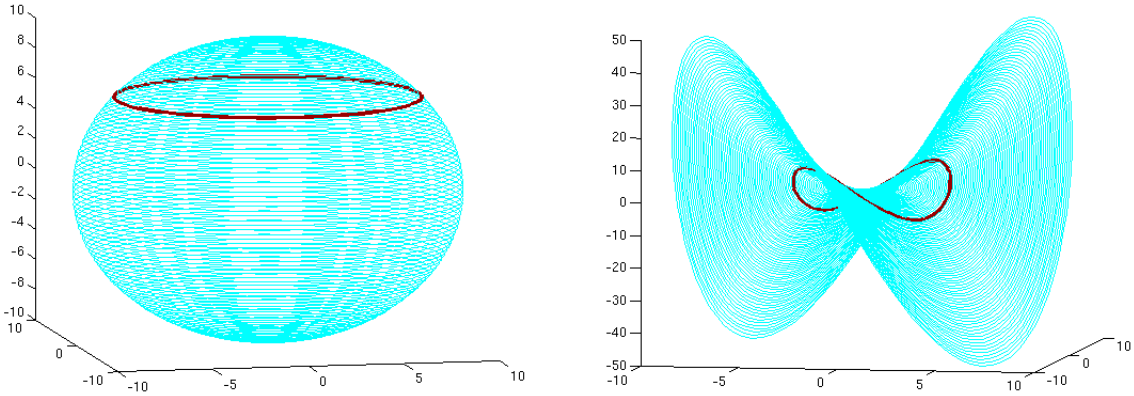


Figure 2.1: **Positively and negatively curved spaces** The radii of the circles are the distances from the center to the circle along the curve. On the sphere, this gives a relatively shorter circumference, while the circumference on the hyperbolic surface is greater than 2πr.

to this point. F may take nonzero values even in flat areas. Calling it (or $\frac{F}{R_i^2}$) the local curvature is therefore somewhat misleading, but it will be called curvature all the same.

The third and last type of curvature which will be encountered in this thesis is the Riemann curvature tensor. It is a tensor of rank $\binom{1}{3}$ and it can be interpreted as a function $TM^3 \rightarrow TM$, where the second and last argument spans a small polygon. The tensor returns the change of its first argument when parallel transported around the polygon to second order in the size scale of the polygon.

One will not dwell over the meaning of this, but present the definition and some useful properties of the Riemann tensor.

The **Riemann tensor** is given by

$$\boldsymbol{R}(\boldsymbol{A}, \boldsymbol{u}, \boldsymbol{v}) = ([\nabla_{\boldsymbol{u}}, \nabla_{\boldsymbol{v}}] - \nabla_{[\boldsymbol{u},\boldsymbol{v}]})\boldsymbol{A}. \tag{2.58}$$

The above expression needs some explanation in order to make sense.

The bracket is the familiar commutator, $[A, B] = AB - BA$. The juxtaposition of to vectors should be read as composition of functions from scalar functions on M to scalar functions on M. That is, composition of differential operators.

The composition of two vectors is not a vector, but the commutator is, as can be seen from

$$[\boldsymbol{u}, \boldsymbol{v}] = (u^\mu v^\nu_{,\mu} - v^\mu u^\nu_{,\mu})\boldsymbol{e}_\nu + (u^\mu v^\nu \frac{\partial^2}{\partial^\mu \partial^\nu} - u^\mu v^\nu \frac{\partial^2}{\partial^\mu \partial^\nu}) \tag{2.59}$$

Similarly the juxtaposition of covariant derivatives are compositions of functions from vector fields to vector fields.

The components are found to be

$$e_\mu R^\mu_{\nu\alpha\beta} = ([\nabla_\alpha, \nabla_\beta] - \nabla_{[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta]})\boldsymbol{e}_\nu. \tag{2.60}$$

As the components are anti-symmetric in the two last indices, one can write the tensor as

$$\boldsymbol{R} = \boldsymbol{R}^\mu_\nu \boldsymbol{e}_\mu \otimes \boldsymbol{\omega}^\nu = R^\mu_{\nu\alpha\beta} \boldsymbol{e}_\mu \otimes \boldsymbol{\omega}^\nu \otimes \boldsymbol{\omega}^\alpha \vee \boldsymbol{\omega}^\beta, \tag{2.61}$$

where $\boldsymbol{R}^\mu_\nu$ can be understood as a matrix of two-forms.

The components of the Riemann curvature tensor can also be expressed with the connection coefficients and the **structure coefficients**, which are defined by

$$[\boldsymbol{e}_\mu, \boldsymbol{e}_\nu] = c^\rho_{\mu\nu} \boldsymbol{e}_\rho. \tag{2.62}$$

This is done by merely inserting the definitions for the covariant derivative (2.48) and the structure coefficients (2.62) into equation (2.60), giving

$$R^\mu_{\nu\alpha\beta} = \Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta} + \Gamma^\rho_{\nu\beta}\Gamma^\mu_{\rho\alpha} - \Gamma^\rho_{\nu\alpha}\Gamma^\mu_{\rho\beta} - c^\rho_{\alpha\beta}\Gamma^\mu_{\nu\rho}. \tag{2.63}$$

This expression shows that the Riemann tensor is an intrinsic property of the manifold, i.e. it is independent of any embedding one may wish to do of the manifold into a higher dimensional space. This is similar to Gauss' *Theorema egregium* which states that Gaussian curvature is intrinsic.

This property of the Riemann tensor is essential for it in order to play its part in general relativity, as the theory only concerns the spacetime manifold and not potentially different embeddings.

The Riemann tensor can be calculated from equation (2.63), however there is a less cumbersome procedure which exploits an anti-symmetry. This procedure is due to Élie Cartan, and is hence referred to as the *Cartan formalism*.

The **connection forms**, $\boldsymbol{\Omega}^\nu_\mu$ can be defined as

$$\boldsymbol{\Omega}^\nu_\mu = \Gamma^\nu_{\mu\alpha}\boldsymbol{\omega}^\alpha. \tag{2.64}$$

They possess an antisymmetry when expressed in an *orthonormal basis*, i.e. a basis where the components of the metric tensor takes the form $g_{\hat{\mu}\hat{\nu}} = diag(-1,\ 1,\ 1,\ 1)$. The indices in such a basis will be tagged with a hat.

The anti-symmetry of the connection forms and equivalently of the connection coefficients are formulated as

$$\boxed{\boldsymbol{\Omega}_{\hat{\mu}\hat{\nu}} = -\boldsymbol{\Omega}_{\hat{\nu}\hat{\mu}}, \quad \Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}} = -\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}}.} \tag{2.65}$$

I will state Cartan's structural equations without proof, a derivation can be found in e.g. [4].

Cartan's first structural equation relates the connection forms to the exterior derivative of basis forms.

$$\boxed{\boldsymbol{d\omega}^{\rho} = -\boldsymbol{\Omega}^{\rho}_{\nu} \vee \boldsymbol{\omega}^{\nu}.} \tag{2.66}$$

Together with the anti-symmetry, this makes it possible to deduce the connection forms from the exterior derivative of the basis forms.

Cartan's second structural equation further relates the connection forms to the Riemann curvature tensor.

$$\boxed{\boldsymbol{R}^{\mu}_{\nu} = \boldsymbol{d\Omega}^{\mu}_{\nu} + \boldsymbol{\Omega}^{\nu}_{\lambda} \vee \boldsymbol{\Omega}^{\lambda}_{\nu}.} \tag{2.67}$$

This suggests a procedure for calculating the Riemann tensor.

(a) Define an orthonormal basis of one-forms, $\boldsymbol{\omega}^{\hat{\mu}}$.

(b) Calculate the exterior derivative of the one-forms, $\boldsymbol{d\omega}^{\hat{\mu}}$.

(c) Deduce the connection forms, $\boldsymbol{\Omega}^{\hat{\mu}}_{\hat{\nu}}$ from equation (2.66).

(d) Calculate the exterior derivative of the connection forms, $\boldsymbol{d\Omega}^{\hat{\mu}}_{\hat{\nu}}$.

(e) Calculate $\boldsymbol{R}^{\hat{\mu}}_{\hat{\nu}}$ by the means of equation (2.67).

This procedure will be carried out in chapter 3.

Furthermore, a symmetric tensor called the **Ricci tensor** is defined as

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}. \tag{2.68}$$

The **Ricci scalar** is the contraction of the Ricci tensor,

$$R = R^{\mu}_{\mu}. \tag{2.69}$$

The **Einstein tensor** is defined as

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{2.70}$$

It can be shown that this tensor is divergence free,

$$E^{\mu\nu}_{;\mu} = 0. \tag{2.71}$$

The Einstein tensor is the geometric entity which appears in Einstein's field equations.

2.2 Special relativity

Before attempting to understand the general theory of relativity, one does wise in spending some time with the special theory of relativity first. As the name suggests, special relativity is a special case of the general theory.

The special theory of relativity was introduced by Albert Einstein in 1905. [2]

2.2.1 Spacetime

In special relativity, spacetime is a four dimensional vector space and elements of this space are called events. The vector space is spanned by a set of basis vectors $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ and any event is written as a linear combination $x^{\mu}\boldsymbol{e}_{\mu}$ for some set of coordinates x^{μ}.

The pseudo inner product between two basis vectors e_{μ} and e_{ν} is defined as

$$\boldsymbol{e}_{\mu} \cdot \boldsymbol{e}_{\nu} = \eta_{\mu\nu} \equiv diag(-1, 1, 1, 1) \tag{2.72}$$

This inner product defines spacetime as **Minkowski space**, which is the simplest example of a Lorentzian manifold. The vector space structure of the manifold itself allows an inner product on the manifold, which is not possible on general manifolds. Note that the metric tensor of the manifold, which is defined as a function from $T_p \times T_p$ to $\mathbb{R}$ for every event p, is $g_{\mu\nu} = diag(-1, 1, 1, 1)$ as well, and the elements of T_p will also be referred to as $\boldsymbol{e}_\mu$. One should have in mind that there is a distinction between the two as they act on different spaces.

When discussing special relativity here, $\eta_{\mu\nu}$ will be used for the position vectors in Minkowski space.

Having introduced the Minkowski space, one would next like to check whether the basis vectors are unique or not when imposing the condition that their pseudo inner product is governed by $\eta_{\mu\nu}$. Not surprisingly, it turns out that there exists a continuum of basis transformations which conserve the form of the pseudo inner product. Such transformations will be referred to as **isometries**.

For example, a set can be chosen according to

$$\begin{array}{c} e_0' = \frac{1}{\sqrt{1-\beta^2}}(e_0 + \beta e_1) \\ e_1' = \frac{1}{\sqrt{1-\beta^2}}(-\beta e_0 + e_1) \\ e_2' = e_2 \\ e_3' = e_3 \end{array} . \tag{2.73}$$

The corresponding coordinate transformations that give $x^\mu e_\mu = x'^\mu e'_\mu$ are seen to be

$$\begin{array}{c} x'^0 = \frac{1}{\sqrt{1-\beta^2}}(x^0 - \beta x^1), \\ x'^1 = \frac{1}{\sqrt{1-\beta^2}}(\beta x^0 + x^1), \\ x'^2 = x^2, \\ x'^3 = x^3. \end{array} \tag{2.74}$$

This shows that Minkowski space is invariant under a certain change of coordinates. This will be seen to correspond to a change of *inertial reference frames.*

In addition to the above change of basis, there are two other types of transformations, which the reader is likely to be familiar with from classical mechanics.
First of all, one may leave the zeroth basis vector as it is, and rotate the spatial vectors.

A pure translation will not leave the lengths of vectors invariant, but lengths of differences of vectors will still be invariant. As the origin (in the sense origin of the coordinate system) of spacetime is chosen arbitrary, this is all we can expect. Also, the important thing is really the metric tensor, which will be unaltered under translations.

The different basis transformations form a mathematical group under composition, and it is referred to as the **Lorentz group**.

Special relativity relates to classical mechanics in the way that x^0 plays the role of time coordinate, while the other three are spatial coordinates. Newtons laws of physics will be correct in the low velocity limit of special relativity when using these coordinates.

2.2.2 The postulates

There are two principles postulated by Einstein [2] that must be taken into account and these make sure that the mechanical laws must be altered when considering objects with higher relative velocities.

The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems in uniform translatory motion relative to each other. (2.75)

The speed of light in vacuum is the same in every inertial reference frame. (2.76)

The above postulates are referred to as "The principle of relativity" and "The principle of invariant light speed" respectively.

These principles can be used to deduce the transformation rules for coordinate transformations between different frames of reference. It turns out that these transformations coincide with the isometries of spacetime. It will merely be demonstrated that the speed of light is invariant and the coordinate-independent Lagrangian mechanics will be presented.

The **four-velocity**, $\boldsymbol{U}(\lambda)$, of a particle moving along a curve $c(\lambda)$ is the element of $T_{c(\lambda)}$ which is the equivalence class of c. If $c(\lambda) = c^\mu(\lambda)\boldsymbol{e}_\mu$, then the four-velocity can be written

$$\boldsymbol{U} = U^\mu \boldsymbol{e}_\mu = \frac{d}{d\lambda}(c^\mu)\boldsymbol{e}_\mu. \tag{2.77}$$

The direction of time can be imposed by demanding that $\frac{d}{d\lambda}(c^0) = U^0 > 0$. This ensures that $c^0(\lambda)$ is injective and the curve might as well be parametrised by c^0. If the curve corresponds to a photon, the absolute value of the three-velocity is $c = 1$[1],

[1] "c" as the speed of light, not the curve. Throughout this thesis, natural units are used.

and this gives

$$(\frac{\partial c^1}{\partial c^0})^2 + (\frac{\partial c^2}{\partial c^0})^2 + (\frac{\partial c^3}{\partial c^0})^2 = 1. \tag{2.78}$$

By the chain rule, this can be written as

$$(U^0)^2 = (U^1)^2 + (U^2)^2 + (U^3)^2, \tag{2.79}$$

but this is just the condition that the length of the four-velocity is zero,

$$\boldsymbol{g}(\boldsymbol{U}, \boldsymbol{U}) = 0. \tag{2.80}$$

This illuminates the reason behind the expression "light-like".

The isometries of $\eta_{\mu\nu}$ are not surprisingly also isometries for $g_{\mu\nu}$. This implies that if the coordinate system is changed, the obedience of the scalar equation (2.80) in any Minkwoskian frame of reference can be expressed as

$$(U^{0'})^2 = (U^{1'})^2 + (U^{2'})^2 + (U^{3'})^2. \tag{2.81}$$

This means that the velocity of light must be equal to 1 in every Minkowskian reference frame.

It remains to introduce the laws of mechanics.

2.2.3 Lagrangian mechanics

As in classical mechanics, the trajectory of particles will be determined from minimizing the action integral,

$$S(c) = \int_c L, \tag{2.82}$$

where c is the curve corresponding to the particle, $c : [0, 1] \to M$ and L is the Lagrangian.

In order for this to be Lorentz invariant, it should be on the form

$$S(c) = \int_0^1 \boldsymbol{\alpha}(\frac{dc}{d\Lambda}(\lambda))d\lambda \tag{2.83}$$

for some one-form $\boldsymbol{\alpha}$. This is the interpretation of the line integral of a one-form. The action for a free particle will take the form

$$S(c) = \int_0^1 -m\sqrt{-\boldsymbol{g}(\frac{dc}{d\lambda}(\lambda), \frac{dc}{d\lambda}(\lambda))}d\lambda. \tag{2.84}$$

This integrand is also linear in $\frac{dc}{d\lambda}(\lambda)$ and can be regarded as a one-form. m is the mass of the particle, assuming it has a mass.

Parametrizing the curve with the time coordinate in some Minkowskian system is always possible for particles traveling at speeds less than or equal to the speed of light. This gives an action on the form

$$S(c) = \int_{t_0}^{t_1} -m\sqrt{1-(\frac{dx}{dt})^2+(\frac{dy}{dt})^2+(\frac{dz}{dt})^2}dt = \int_{t_0}^{t_1} -m\sqrt{1-\beta^2}dt. \qquad (2.85)$$

Here, the interpretation of β is introduced. It is the three-velocity of a particle divided by the speed of light (which is normalised to 1 here). It also becomes apparent that a photon must have a different Lagrangian as it is massless and has $1-\beta^2=0$.

Imposing Lagrange's equations on the Lagrangian, $L=-m\sqrt{1-\beta^2}$ gives

$$\frac{d}{dt}(\frac{\dot{x}^i}{\sqrt{1-\beta^2}}) = 0 \implies \frac{d}{dt}(\frac{\beta^2}{1-\beta^2}) = 0 \implies \ddot{x}^i = 0. \qquad (2.86)$$

This means that a freely moving particle is moving along a straight line.

Note that the except for the factor $-m$, the integral is the proper time of the particle, introduced in subsection 2.1.7. A massive, free particle follows the curve with maximal proper time.

As mentioned above, (2.74) is a coordinate transformation between inertial reference frames. An **inertial reference frame** is a choice of coordinates which gives a Minkowskian metric. To be more precise, different coordinate systems which are related by a spatial rotation will be considered as the same inertial reference frame.

As free particles move along straight lines, one possible solution is that a particle has constant spatial coordinates in some inertial reference frame. Such a reference frame is called **the rest frame of the particle**.

Suppose a particle is moving with a velocity $-\beta$ in the x^1 direction in the coordinate system $\{x^0,x^1,x^2,x^3\}$. This can be parametrised as

$$x^0=x^0,\ x^1=(-\beta x^0+x_0^1),\ x^2=x_0^2,\ x^3=x_0^3 \qquad (2.87)$$

Imposing transformation (2.74) gives

$$\begin{aligned} x'^0 &= \gamma((1+\beta)x^0-\beta x_0^1), \\ x'^1 &= \gamma(x_0^1), \\ x'^2 &= x_0^2, \\ x'^3 &= x_0^3. \end{aligned} \qquad (2.88)$$

Here, the **Lorentz factor**, $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$ was introduced. The spatial coordinates are constant in the primed coordinate system. This shows that the coordinate transformation was a transformation to the rest frame of the particle.

Expanding the Lagrangian in a Taylor series to second order shows that the free particle Lagrangian from classical mechanics is obtained in the low velocity limit.

$$L = -m\sqrt{1-\beta^2} \approx -m + \frac{1}{2}m\beta^2 = -mc^2 + \frac{1}{2}mv^2 = T - V \tag{2.89}$$

In order to include interactions, the Lagrangian must be modified in a way that preserves Lorentz invariance. This invariance is for example responsible for unifying the electric and magnetic forces in the electromagnetic theory. Gravity will be included in the general theory of relativity.

2.3 The General Theory of Relativity

Einstein published his general theory of relativity in 1916[3]. In general relativity, gravity is incorporated as a geometric effect. The geometry of spacetime depends on the distribution of matter and matter is moving according to the geometry of spacetime. Particles moving under influence of no other forces than gravity are said to be free. In fact, when this is established, gravity is no longer considered a force at all.

2.3.1 The principle of equivalence

There is a peculiar feature of gravity which distinguishes it from other forces. The acceleration that is experienced by an object due to gravitational pull from a second object is independent of the mass of the first object. This is because the inertial mass from Newtons second law coincides with the gravitational mass in Newtons law of gravity.

This means that every constituent of an object at rest in a uniform gravitational field is acted upon by a force which would have caused a constant acceleration if there were no gravitational field.

Consider a box that travels in free space, at same constant acceleration as the before mentioned object would have experienced if it was freely falling. Suppose the box contains an object which is at rest relative to the box. The forces that act on the object are completely equivalent to the forces acting on the object in the gravitational field.

A person, who's knowledge of this world is limited to the inside of the box, can not determine whether the box is accelerating or situated in a gravitational field.

The principle of equivalence states that *the local behaviour of matter in an accelerated reference frame can not be distinguished from the behaviour in a gravitational field.*

This means that gravity is no longer considered a force. Freely falling particles in a gravitational field are not acted upon by forces, completely equivalent to particles moving in empty space. In the case of an inhomogeneous gravitational field, one may introduce a *local inertial reference frame* following the freely falling particle.

If this is going to make sense, the seemingly curved trajectories of falling bodies must be straight lines or *geodesics* in the spacetime manifold. In order to achieve this, spacetime must be equipped with a metric which is not Minkowskian.

2.3.2 Spacetime and geodesics

The spacetime of general relativity is a Lorentzian manifold. That is, it is a differentiable manifold equipped with a metric with signature $\{-,+,+,+\}$, at least in four-dimensional theories. Similar to the special theory, the trajectories of free particles are determined from the metric by minimizing the action integral

$$S(c) = \int_0^1 -m\sqrt{-\boldsymbol{g}(\frac{dc}{d\lambda}(\lambda), \frac{dc}{d\lambda}(\lambda))}d\lambda. \tag{2.90}$$

The big difference is that the metric tensor may be an arbitrary tensor as long as it is symmetric and has the right signature. Note that as opposed to the special theory of relativity where the whole spacetime is homeomorphic to $\mathbb{R}^4$, the spacetime of general relativity can take any orientable shape.

Denote the four-velocity of the particle by $\frac{dc}{d\lambda}(\lambda) = \boldsymbol{u} = u^\mu \boldsymbol{e}_\mu$. In special relativity, Lagrange's equations give the relation that

$$\frac{d}{d\lambda}(\boldsymbol{u}) = \dot{\boldsymbol{u}} = u^\mu_{,\nu} u^\nu \boldsymbol{e}_\mu = 0. \tag{2.91}$$

The four-velocity is constant, giving straight lines as solutions.

It can be shown that in general relativity, minimizing the action gives a very similar relation,

$$\boxed{\frac{d}{d\lambda}(\boldsymbol{u}) = \dot{\boldsymbol{u}} = u^\mu_{;\nu} u^\nu \boldsymbol{e}_\mu = 0.} \tag{2.92}$$

A **geodetic curve** or a **geodesic** is a curve with a tangent vector satisfying the relation (2.92). These are the paths of free particles. As the equation which governs the trajectories of free particles ought to reduce to (2.91) in the special theory, this is not very surprising. As discussed previously, the partial derivative of equation (2.91) does not transform as a tensor. However, when restricted to the coordinate systems with Minkowskian metric, the Christoffel symbols vanishes and this makes (2.91) valid in the special theory.

This establishes the way free particles move in spacetime, but as gravity depends on energy distributions and the metric of spacetime is acting as gravity, there must be a way to determine the dynamics of the metric as well.

2.3.3 Einstein's field equations

If the metric is to be related to the contents of the universe in any way, there must be some mathematical entity, preferably a tensor, that describes the matter content. The conservation of energy and momentum can be incorporated as the vanishing divergence of a tensor of rank 2, and it is in fact incorporated in this way.

The **stress-energy-momentum tensor**, or **stress-energy tensor/ energy-momentum tensor** is a symmetric tensor of rank 2. It has vanishing divergence,

$$\boxed{T^{\mu\nu}_{;\nu} = 0} \tag{2.93}$$

and its content depends on the content of spacetime. In the simplest cases, it can be written as

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{2.94}$$

where ρ is the energy density, p is the pressure and u^μ are the components of the four velocity field.

Einsteins field equations are collected in the single tensor equation[2].

$$\boxed{E_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}.} \tag{2.95}$$

The Einstein tensor was introduced in subsection 2.1.10 and the constant Λ is the **cosmological constant**. It is an integration constant which emerges in the derivation of the field equations. Although it emerges as a mathematical entity, it can also be absorbed into the energy-momentum tensor and be interpreted as

[2]As wel as c, $\kappa = \frac{8\pi G}{c^4} \equiv 1$ in natural units.

Lorentz invariant vacuum energy, LIVE for short. The vanishing divergence of the energy-momentum tensor is seen as both tensors on the left hand side of equation (2.95) have zero divergence.

The derivation of the field equations will be omitted here, actually it has been asserted that no *rigorous* derivation may be given[1].

The derivations that nevertheless are given are based on coordinate independence, the principle of equivalence and reduction to Newtonian gravity in the low energy limit.

The field equations may be applied in several ways. On planetary scale, general relativity must be applied in order to obtain the necessary precision in measurements of distances and time differences for e.g. GPS. On slightly larger scale, e.g. the spacetime geometry around black holes can be found. It is on even larger scales though, that the theory will be applied in this thesis.

2.3.4 Cosmology

Cosmology is the study of the dynamics of the universe as a whole. Before the introduction of general relativity in 1916, the universe was believed to be static. General relativity introduces the possibility of a dynamic universe. For example, the universe may expand or contract, leading to respectively more or less available space for particles to reside in. This also causes distances between particles to change at a rate proportional to the distance.

When considering the evolution of the universe at a very large scale, small variations in mass density over short distances will not be important. The energy-momentum tensor at a point will of course be very dependent of whether the point is in empty space, a planet or possibly a star or a black hole. As the scale of interest is even larger than galactic clusters, these fluctuations in the energy can effectively be smeared out.

Under this approximation, there are two principles which are often assumed, called **the cosmological principles**.

Homogeneity: *There is no special point in the universe. Matter is evenly distributed in the universe at large scales.*

Isotropy: *There is no special spatial direction in the universe. The expansion is the same in all directions.*

These principles give huge restrictions on the metric, giving the Friedmann-Robertson-Walker metric.

$$ds^2 = -dt^2 + a^2(t)(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2) \tag{2.96}$$

Imposing Einsteins equation on this metric and using the form (2.94) for the energy-momentum tensor gives the Friedmann equations

$$3\frac{\dot{a}^2 + k}{a^2} = \rho \tag{2.97}$$

and

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} = p. \tag{2.98}$$

A further restriction can be obtained by looking at the relations between energy density and pressure for different fluids (read matter). A **perfect fluid** is a fluid which obeys the relation

$$\boxed{p = \omega\rho,} \tag{2.99}$$

where ω is some constant. For example, radiation obeys (2.99) with $\omega = \frac{1}{3}$ while dust (or material particles) has $\omega = 0$ at low temperatures (mc$^2 \gg$ kT).

As the title of this thesis implies, I will treat inhomogeneous models in this thesis. This breaks with the cosmological principles and in order to still be in accordance with isotropic observations, the metric will be assumed to be spherically symmetric. This is the case for the Lemaître-Tolman-Bondi metric (LTB for short), which is the subject of the next chapter.

Chapter 3

The Lemaître-Tolman-Bondi metric

$$ds^2 = -dt^2 + X^2(r,t)dr^2 + R^2(r,t)(d\theta^2 + sin^2\theta d\phi^2) \tag{3.1}$$

3.1 Calculating the Einstein Tensor using Cartan's formalism

The Cartan formalism was outlined in subsection 2.1.10

An orthonormal basis of one-forms is found from (3.1),

$$\boldsymbol{\omega}^{\hat{t}} = dt,\ \boldsymbol{\omega}^{\hat{r}} = Xdr,\ \boldsymbol{\omega}^{\hat{\theta}} = Rd\theta,\ \boldsymbol{\omega}^{\hat{\phi}} = Rsin(\theta)d\phi. \tag{3.2}$$

Their exterior derivatives are

$$\boldsymbol{d\omega}^{\hat{t}} = 0, \tag{3.3}$$

$$\boldsymbol{d\omega}^{\hat{r}} = \frac{\dot{X}}{X}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{r}}, \tag{3.4}$$

$$\boldsymbol{d\omega}^{\hat{\theta}} = \frac{\dot{R}}{R}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\theta}} + \frac{R'}{XR}\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\theta}}, \tag{3.5}$$

$$\boldsymbol{d\omega}^{\hat{\phi}} = \frac{\dot{R}}{R}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\phi}} + \frac{R'}{XR}\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\phi}} + \frac{cot(\theta)}{R}\boldsymbol{\omega}^{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\phi}}. \tag{3.6}$$

Cartan's first structure equation (2.66) or (3.7) introduces the connection forms $\boldsymbol{\Omega}^{\rho}_{\nu}$.

$$\boxed{\boldsymbol{d\omega}^{\rho} = -\boldsymbol{\Omega}^{\rho}_{\nu} \wedge \boldsymbol{\omega}^{\nu}} \tag{3.7}$$

The connection forms are found to be

$$\boldsymbol{\Omega}^{\hat{r}}_{\hat{t}} = \boldsymbol{\Omega}^{\hat{t}}_{\hat{r}} = \frac{\dot{X}}{X}\boldsymbol{\omega}^{\hat{r}}, \tag{3.8}$$

$$\boldsymbol{\Omega}^{\hat{\theta}}_{\hat{t}} = \boldsymbol{\Omega}^{\hat{t}}_{\hat{\theta}} = \frac{\dot{R}}{R}\boldsymbol{\omega}^{\hat{\theta}}, \tag{3.9}$$

$$\boldsymbol{\Omega}^{\hat{\theta}}_{\hat{r}} = -\boldsymbol{\Omega}^{\hat{r}}_{\hat{\theta}} = \frac{R'}{XR}\boldsymbol{\omega}^{\hat{\theta}}, \tag{3.10}$$

$$\boldsymbol{\Omega}^{\hat{\phi}}_{\hat{t}} = \boldsymbol{\Omega}^{\hat{t}}_{\hat{\phi}} = \frac{\dot{R}}{R}\boldsymbol{\omega}^{\hat{\phi}}, \tag{3.11}$$

$$\boldsymbol{\Omega}^{\hat{\phi}}_{\hat{r}} = -\boldsymbol{\Omega}^{\hat{r}}_{\hat{\phi}} = \frac{R'}{XR}\boldsymbol{\omega}^{\hat{\phi}}, \tag{3.12}$$

and

$$\boldsymbol{\Omega}^{\hat{\phi}}_{\hat{\theta}} = -\boldsymbol{\Omega}^{\hat{\theta}}_{\hat{\phi}} = \frac{cot\theta}{R}\boldsymbol{\omega}^{\hat{\phi}}. \tag{3.13}$$

While the exterior derivatives are

$$\boldsymbol{d\Omega}^{\hat{r}}_{\hat{t}} = \boldsymbol{d\Omega}^{\hat{t}}_{\hat{r}} = \frac{\ddot{X}}{X}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{r}}, \tag{3.14}$$

$$\boldsymbol{d\Omega}^{\hat{\theta}}_{\hat{t}} = \boldsymbol{d\Omega}^{\hat{t}}_{\hat{\theta}} = \frac{\ddot{R}}{R}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\theta}} + \frac{\dot{R}'}{XR}\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\theta}}, \tag{3.15}$$

$$\boldsymbol{d\Omega}^{\hat{\theta}}_{\hat{r}} = -\boldsymbol{d\Omega}^{\hat{r}}_{\hat{\theta}} = \frac{\dot{R}'X - R'\dot{X}}{X^2R}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\theta}} + \frac{R'' - R'X'}{X^3R}\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\theta}}, \tag{3.16}$$

$$\boldsymbol{d\Omega}^{\hat{\phi}}_{\hat{t}} = \boldsymbol{d\Omega}^{\hat{t}}_{\hat{\phi}} = \frac{\ddot{R}}{R}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\phi}} + \frac{\dot{R}'}{RX}\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\phi}} + \frac{\dot{R}cot\theta}{R^2}\boldsymbol{\omega}^{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\phi}}, \tag{3.17}$$

$$\boldsymbol{d\Omega}^{\hat{\phi}}_{\hat{r}} = -\boldsymbol{d\Omega}^{\hat{r}}_{\hat{\phi}} = \frac{\dot{R}'X - R'\dot{X}}{X^2R}\boldsymbol{\omega}^{\hat{t}}\wedge\boldsymbol{\omega}^{\hat{\phi}} + \frac{R''X - R'X'}{X^3R}\boldsymbol{\omega}^{\hat{r}}\wedge\boldsymbol{\omega}^{\hat{\phi}} + \frac{R'cot\theta}{XR^2}\boldsymbol{\omega}^{\hat{\theta}}\wedge\boldsymbol{\omega}^{\hat{\phi}}, \tag{3.18}$$

and

$$\boldsymbol{d\Omega}^{\hat{\phi}}_{\hat{\theta}} = -\boldsymbol{d\Omega}^{\hat{\theta}}_{\hat{\phi}} = \frac{-1}{R^2}\boldsymbol{\omega}^{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\phi}}. \tag{3.19}$$

Cartan's second structure equation (3.20) or (2.67) relates the Riemann curvature tensor to the connection forms,

$$\boxed{\boldsymbol{R}^{\hat{\mu}}_{\hat{\nu}} = \boldsymbol{d\Omega}^{\hat{\mu}}_{\hat{\nu}} + \boldsymbol{\Omega}^{\hat{\mu}}_{\hat{\lambda}} \wedge \boldsymbol{\Omega}^{\hat{\lambda}}_{\hat{\nu}}.} \tag{3.20}$$

This gives for the Riemann tensor,

$$\boldsymbol{R}^{\hat{r}}_{\hat{t}} = \boldsymbol{R}^{\hat{t}}_{\hat{r}} = \frac{\ddot{X}}{X}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{r}}, \tag{3.21}$$

$$\boldsymbol{R}^{\hat{\theta}}_{\hat{t}} = \boldsymbol{R}^{\hat{t}}_{\hat{\theta}} = \frac{\ddot{R}}{R}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\theta}} + \frac{\dot{R}'X - R'\dot{X}}{X^2R}\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\theta}}, \tag{3.22}$$

$$\boldsymbol{R}^{\hat{\theta}}_{\hat{r}} = -\boldsymbol{R}^{\hat{r}}_{\hat{\theta}} = \frac{\dot{R}'X - R'\dot{X}}{X^2R}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\theta}} + \frac{R''X - R'X''\dot{R}\dot{X}X^2}{X^3R}\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\theta}}, \tag{3.23}$$

$$\boldsymbol{R}^{\hat{\phi}}_{\hat{t}} = \boldsymbol{R}^{\hat{t}}_{\hat{\phi}} = \frac{\ddot{R}}{R}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\phi}} + \frac{\dot{R}'X - R'\dot{X}}{X^2R}\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\phi}}, \tag{3.24}$$

$$\boldsymbol{R}^{\hat{\phi}}_{\hat{r}} = -\boldsymbol{R}^{\hat{r}}_{\hat{\phi}} = \frac{\dot{R}'X - R'\dot{X}}{X^2R}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\phi}} + \frac{R''X - R'X''\dot{R}\dot{X}X^2}{X^3R}\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\phi}} \tag{3.25}$$

and

$$\boldsymbol{R}^{\hat{\phi}}_{\hat{\theta}} = -\boldsymbol{R}^{\hat{\theta}}_{\hat{\phi}} = \frac{R'^2 - \dot{R}^2X^2 - X^2}{R^2X^2}\boldsymbol{\omega}^{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\phi}}. \tag{3.26}$$

The non zero components of the Ricci tensor are:

$$R_{\hat{t}\hat{t}} = \frac{-\ddot{X}}{X} - 2\frac{\ddot{R}}{R}, \tag{3.27}$$

$$R_{\hat{r}\hat{r}} = \frac{\ddot{X}}{X} + 2\frac{R'X' + \dot{R}\dot{X}X^2 - R''X}{RX^3}, \tag{3.28}$$

$$R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}} = \frac{\ddot{R}}{R} + \frac{R'X' + \dot{R}\dot{X}X^2 - R''X}{RX^3} + \frac{\dot{R}^2 + 1}{R^2} - \frac{R'^2}{R^2X^2}, \tag{3.29}$$

$$R_{\hat{t}\hat{r}} = R_{\hat{r}\hat{t}} = 2\frac{R'\dot{X} - \dot{R}'X}{RX^2} \tag{3.30}$$

and

$$R = R^{\hat{\mu}}_{\hat{\mu}} = 2\frac{\ddot{X}}{X} + 4\frac{\ddot{R}}{R} + 4\frac{R'X' + \dot{R}\dot{X}X^2 - R''X}{RX^3} + 2\frac{\dot{R}^2 + 1}{R^2} - 2\frac{R'^2}{R^2X^2}. \tag{3.31}$$

The Einstein tensor, $E_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \frac{1}{2}R\eta_{\hat{\mu}\hat{\nu}}$ has these non zero components:

$$E_{\hat{t}\hat{t}} = 2\frac{R'X' + \dot{R}\dot{X}X^2 - R''X}{RX^3} + \frac{\dot{R}^2+1}{R^2} - \frac{R'^2}{R^2X^2} \tag{3.32}$$

$$E_{\hat{r}\hat{r}} = -2\frac{\ddot{R}}{R} - \frac{\dot{R}^2+1}{R^2} + \frac{R'^2}{R^2X^2} \tag{3.33}$$

$$E_{\hat{\theta}\hat{\theta}} = E_{\hat{\phi}\hat{\phi}} = -\frac{\ddot{X}}{X} - \frac{\ddot{R}}{R} - \frac{R'X' + \dot{R}\dot{X}X^2 - R''X}{RX^3} \tag{3.34}$$

$$E_{\hat{t}\hat{r}} = E_{\hat{r}\hat{t}} = 2\frac{R'\dot{X} - \dot{R}'X}{RX^2} \tag{3.35}$$

3.2 Choosing coordinates

The metric (3.1) can be simplified by making choices about the coordinates on spacetime.

3.2.1 Radial distance coordinates

As an example, one can let the radial coordinate be equal to the spatial distance from the origin. This is equivalent to the condition $X \equiv 1$ and the metric takes the simpler form (3.36).

$$ds^2 = -dt^2 + dr^2 + R^2(r,t)(d\theta^2 + sin^2\theta d\phi^2) \tag{3.36}$$

A serious drawback with this choice is that the world lines of an average freely falling particle will have both radial- and time components that depend on the distance to the origin (except in static cases) and the energy-momentum tensor becomes complicated.

3.2.2 Comoving coordinates

An other choice is to use comoving coordinates. This corresponds to particles having constant spatial coordinates on average in areas on the scale of clusters of galaxies. This assumption sets the spatial components of the four velocity field to zero. This makes the energy-momentum tensor diagonal, and thus also the Einstein tensor which leads to the condition (3.37)

$$E_{\hat{t}\hat{r}} = E_{\hat{r}\hat{t}} = 2\frac{R'\dot{X} - \dot{R}'X}{RX^2} = 0 \tag{3.37}$$

X must be nonzero everywhere and R is nonzero except for r=0[1], and possibly one other point, when the coordinates make little sense anyway. This means that one has the condition (3.38)

$$R'\dot{X} = \dot{R}'X \tag{3.38}$$

The equation (3.38) can be treated in two separate cases, areas of spacetime where $R' \neq 0$ and areas where $R' = 0$.

If $R' \neq 0$ one has

$$\frac{\dot{X}}{X} = \frac{\dot{R}'}{R'}, \tag{3.39}$$

which can be integrated and gives

$$X(r,t) = R'(r,t)f(r), \tag{3.40}$$

where $f(r)$ is some function of the radial coordinate only. This leads to the LTB metric with comoving coordinates (3.41), where $F \equiv 1 - \frac{1}{f}$.

$$\boxed{ds^2 = -dt^2 + \frac{R'^2(r,t)}{1-F(r)}dr^2 + R^2(r,t)(d\theta^2 + sin^2\theta d\phi^2)} \tag{3.41}$$

Now the Einstein tensor has the non-zero components shown in eqs. (3.42-3.44)

$$E_{\hat{t}\hat{t}} = \frac{2\dot{R}\dot{R}'R + R'(\dot{R}^2 + F) + F'R}{R'R^2} \tag{3.42}$$

$$E_{\hat{r}\hat{r}} = -\frac{2\ddot{R}R + \dot{R}^2 + F}{R^2} \tag{3.43}$$

$$E_{\hat{\theta}\hat{\theta}} = E_{\hat{\phi}\hat{\phi}} = -\frac{\frac{1}{2}F'R + \ddot{R}'R^2 + \ddot{R}RR' + \dot{R}\dot{R}'R}{R^2R'} \tag{3.44}$$

It is apparent that this breaks down in the case where $R' = 0$ as the radial coordinate becomes stationary. Such cases necessarily appear if a spatial section of

[1] At least if we choose the origin at r=0, which is natural, but not necessary

spacetime is geometrically closed.

If $R' = 0$, (3.38) becomes

$$\dot{R}'X = 0 \Leftrightarrow \dot{R}' = 0 \tag{3.45}$$

Thus one obtains the interesting result that

$$\exists\ t_0 \mid R'(r,t_0) = 0 \implies R'(r,t) = 0\ \ \forall t \tag{3.46}$$

The Einstein tensor looks a lot simpler in these cases.

$$E_{\hat{t}\hat{t}} = 2\frac{\dot{R}\dot{X}X^2 - R''X}{RX^3} + \frac{\dot{R}^2+1}{R^2} \tag{3.47}$$

$$E_{\hat{r}\hat{r}} = -2\frac{\ddot{R}}{R} - \frac{\dot{R}^2+1}{R^2} \tag{3.48}$$

$$E_{\hat{\theta}\hat{\theta}} = E_{\hat{\phi}\hat{\phi}} = -\frac{\ddot{X}}{X} - \frac{\ddot{R}}{R} - \frac{\dot{R}\dot{X}X^2 - R''X}{RX^3} \tag{3.49}$$

In the case of intervals of r where $R' = 0$, the condition $R'' = 0$ becomes valid as well. If $R' = 0$ and $R'' \neq 0$, this problem is solved simply by solving the equations for $R' \neq 0$ and finding X and R at $R' = 0$ by calculating the limits.

3.3 Covariant decomposition

In order to find the covariant decomposition one has to find the covariant derivative of the four velocity field with components u^μ.

With comoving coordinates, the four velocity can be written as

$$u^\mu = \delta^\mu_t. \tag{3.50}$$

The covariant derivative of the components of a vector field, A^μ is defined in (2.48)

$$A^\mu_{;\nu} \equiv A^\mu_{,\nu} + A^\alpha \Gamma^\mu_{\alpha\nu}, \tag{3.51}$$

where the Γ's are the Christoffel symbols. They can be calculated from the metric tensor (2.49) as

$$\Gamma^{\mu}_{\alpha\nu} = \frac{1}{2} g^{\mu\lambda}(g_{\lambda\alpha,\nu} + g_{\lambda\nu,\alpha} - g_{\alpha\nu,\lambda}) \tag{3.52}$$

and as can be seen, they are symmetric in the lower indices.

For comoving coordinates, only the Christoffel symbols $\Gamma^{\mu}_{t\nu}$ plays a role when calculating the covariant derivative of the four velocity. These are found to be

$$\Gamma^{r}_{tr} = \frac{\dot{X}}{X} = \frac{\dot{R}'}{R'}, \quad \Gamma^{\theta}_{t\theta} = \Gamma^{\phi}_{t\phi} = \frac{\dot{R}}{R} \tag{3.53}$$

It follows that $u^{\mu}_{;\nu}$ is diagonal with the components shown in (3.54)

$$u^{t}_{;t} = 0, \ u^{r}_{;r} = \frac{\dot{R}'}{R'}, \ u^{\theta}_{;\theta} = u^{\phi}_{;\phi} = \frac{\dot{R}}{R} \tag{3.54}$$

The expansion scalar is found to be

$$\theta = u^{\mu}_{;\mu} = \frac{\dot{R}'}{R'} + 2\frac{\dot{R}}{R}. \tag{3.55}$$

The shear tensor is

$$\sigma^{\alpha}_{\beta} = diag(0, \ -2\sigma, \ \sigma, \ \sigma), \quad \sigma \equiv \frac{1}{3}(\frac{\dot{R}}{R} - \frac{\dot{R}'}{R'}) \tag{3.56}$$

While the anti-symmetric vorticity tensor is of course 0.

Chapter 4

Radiation dominated universe with Λ

The Einstein field equations have been solved[8] for LTB universes with dust and a cosmological constant. Inspired by this article, one would like to do something similar with universes containing only radiation and a cosmological constant.

This task is carried out in this chapter. The exact analytical solution for LTB universes with radiation and cosmological constant will be derived in a way that much resembles the derivation of the dust dominated model in [8].

However, it turns out that radiation dominated universes distinguishes itself from dust dominated universes in a critical way. Dissimilar to the dust dominated LTB universe, radiation dominated LTB universes must be homogeneous. In the course of study of this model, this became apparent to the author after having found a general inhomogeneous solution.

The presentation of the analysis will follow a chronological order, thus deriving the general inhomogeneous solution before showing that it must reduce to the homogeneous case. Even though the solutions will simplify in the end, the method that was used to derive the inhomogeneous solution can be interesting in its own right.

4.1 Field equations

The energy-momentum tensor of an electromagnetic field can be written

$$T_{\mu\nu} = diag[\rho(t,r), \frac{\rho(t,r)}{3}, \frac{\rho(t,r)}{3}, \frac{\rho(t,r)}{3}] \tag{4.1}$$

Recall the form of the Einstein tensor for $R' \neq 0$ in (3.42-3.44).

$$(E_{\hat{t}\hat{t}} - \Lambda)R^2R' = [R(\dot{R}^2 + F - \frac{\Lambda}{3}R^2)]' = \rho R^2 R'. \tag{4.2}$$

Using that the trace of the energy-momentum tensor is zero gives

$$-\frac{1}{2}R^2R'(E^\mu_\mu + 4\Lambda) = [R(\dot{R}^2 + F) - \frac{2\Lambda}{3}R^3 + \ddot{R}R^2]' = 0. \tag{4.3}$$

Also,

$$E_{\hat{r}\hat{r}} - E_{\hat{\theta}\hat{\theta}} = 0 \implies \left[\frac{R(\dot{R}^2 + F) + 2R^2R'}{R^3}\right]' = 0. \tag{4.4}$$

Integration of the above combinations of the Einstein equations gives (4.5-4.7)

It is apparent that $M(t, 0) = 0 \;\forall\; t$. Also, the left hand side of (4.6) must be zero for $r = 0$. $F(0) = 0$ if the metric is differentiable at $r = 0$ and as $R(t, 0) \equiv 0 \forall t$ also the time derivatives up to arbitrary order must be zero at $r = 0$. As the right hand side must be a function of t only, it must be zero.

$$\int_0^r \rho(t,q)R^2(t,q)R'(t,q)dq \equiv M(t,r) = R(\dot{R}^2 + F - \frac{\Lambda}{3}R^2) \tag{4.5}$$

$$R(\dot{R}^2 + F) - \frac{2\Lambda}{3}R^3 + \ddot{R}R^2 = 0 \tag{4.6}$$

$$\frac{R(\dot{R}^2 + F) + 2R^2\ddot{R}}{R^3} = \beta(t) \tag{4.7}$$

The time derivative of M is found to be

$$\dot{M} = \dot{R}(\dot{R}^2 + F - \Lambda R^2 + 2\ddot{R}R) = -\dot{R}R^2(E_{\hat{r}\hat{r}} + \Lambda) = -\frac{\rho}{3}\dot{R}R^2. \tag{4.8}$$

In the last equality, the radial component Einstein equation was utilized ($E_{\hat{r}\hat{r}} + \Lambda = T_{\hat{r}\hat{r}}$).

$$M' = \rho R' R^2 \tag{4.9}$$

If also $\rho \neq 0$, then

$$\frac{\dot{M}}{M'} = -\frac{1}{3}\frac{\dot{R}}{R'} \tag{4.10}$$

Combining (4.5-4.7) and solving for $\ddot{R}$ leads to

$$\ddot{R} = (\beta - \frac{2}{3}\Lambda)R = -\frac{M}{R^2} + \frac{\Lambda}{3}R \tag{4.11}$$

and

$$\frac{M}{R^3} = -(\beta + \Lambda). \tag{4.12}$$

The right hand side is a function of time, only, so

$$\left(\frac{M}{R^3}\right)' = \frac{M'}{R^3} - 3\frac{MR'}{R^4} = 0 \tag{4.13}$$

Further assuming $M \neq 0$ ($\Leftrightarrow M' \neq 0$ since $R \neq 0 \neq R'$ is already assumed)

$$\frac{M}{M'} = \frac{1}{3}\frac{R}{R'} \tag{4.14}$$

Combining (4.10) and (4.14) leads to

$$\frac{\dot{M}}{M} = -\frac{\dot{R}}{R}, \tag{4.15}$$

$$(\dot{MR}) = \dot{M}R + \frac{M\dot{R}}{R^2} = 0, \tag{4.16}$$

and I define

$$MR \equiv f(r). \tag{4.17}$$

Note Satisfied with having found a combination of R and M which is time independent, the derivation from here will continue very similar to the derivation for dust dominated models found in [8].

Studying the equations (4.10, 4.14, 4.15), one observes that any one of them follows from the other two. However, equation (4.15) does not contain all the information of the other two. From here, one should have utilized both equations (4.14-4.15) and obtained the homogeneity straight away.

Equation (4.14) is going to be the basis for section 4.3.

Combining equations (4.5) and (4.17) leads to (4.18) which can be solved when given initial conditions (4.19)

$$\dot{R}^2 = \frac{f}{R^2} - F + \frac{\Lambda}{3}R^2 \tag{4.18}$$

$$R(t_i, r) = R_i(r), \quad M(t_i, r) = \frac{f(r)}{R_i(r)}, \quad F(r) \tag{4.19}$$

4.2 Analytic solution

This section is dedicated to solving the differential equation (4.18).

Changing variable function to $u = R^2$ gives

$$\dot{u}^2 = 4u\dot{R}^2 = \frac{4\Lambda}{3}u^2 - 4Fu + 4f. \tag{4.20}$$

Equation (4.20) can be solved in two separate cases, $\Lambda \neq 0$ and $\Lambda = 0$.

Taking the simplest case first, in the case $\Lambda = 0$, (4.20) simplifies to

$$\dot{u}^2 = 4(f - Fu), \tag{4.21}$$

$$\frac{du}{\sqrt{\frac{f}{F} - u}} = \pm 2\sqrt{F}dt, \tag{4.22}$$

$$\sqrt{\frac{f}{F} - R^2} = \mp\sqrt{F}(t - t_i) + \sqrt{\frac{f}{F} - R_i^2} \tag{4.23}$$

and finally the solution is

$$R^2 = R_i^2 - F(t - t_i)^2 \mp 2\sqrt{f - FR_i^2}(t - t_i). \tag{4.24}$$

If $\Lambda \neq 0$ one can write

$$\dot{u}^2 = \frac{4\Lambda}{3}((u - \frac{3F}{2\Lambda})^2 + \frac{3f}{\Lambda} - \frac{9F^2}{4\Lambda^2}) \tag{4.25}$$

or

$$\dot{v}^2 = \frac{4\Lambda}{3}(v^2 + k) \tag{4.26}$$

where $v = u - \frac{3F}{2\Lambda}$ and $k = \frac{3f}{\Lambda} - \frac{9F^2}{4\Lambda^2}$.

If $k = 0$, then

$$\dot{v} = \pm\sqrt{\frac{4\lambda}{3}}v \tag{4.27}$$

and R^2 is found to be

$$\boxed{R^2 = (R_i^2 - \frac{3F}{2\Lambda})e^{\pm\sqrt{\frac{4\lambda}{3}}(t-t_i)} + \frac{3F}{2\Lambda}.} \tag{4.28}$$

Assuming $k \neq 0$ and setting $w = \frac{v}{\sqrt{k}}$ gives

$$\frac{dw}{\sqrt{1+w^2}} = \pm\sqrt{\frac{4\Lambda}{3}}dt. \tag{4.29}$$

The integral can be expressed as

$$\frac{R^2 - \frac{3F}{2\Lambda} + \sqrt{R^4 - \frac{3F}{\Lambda}R^2 + \frac{3f}{\Lambda}}}{R_i^2 - \frac{3F}{2\Lambda} + \sqrt{R_i^4 - \frac{3F}{\Lambda}R_i^2 + \frac{3f}{\Lambda}}} = e^{\pm\sqrt{\frac{4\Lambda}{3}}(t-t_i)} \tag{4.30}$$

or solved for R^2 as

$$R^2 = \sqrt{\frac{3f}{\Lambda} - \frac{9F^2}{4\Lambda^2}}sinh(\pm\sqrt{\frac{4\Lambda}{3}}(t-t_i) + arsinh(w_i)) + \frac{3F}{2\Lambda}. \tag{4.31}$$

It is evident that if $k = \frac{3f}{\Lambda} - \frac{9F^2}{\Lambda^2} > 0$, then the solution will be real and either start at $-\infty$ and reach $R^2 = 0$ at a finite point in time t, or it will start out from $R^2 = 0$ and increase for ever. Also, if $\Lambda < 0$, the solution will be complex, so $k > 0 \implies \Lambda \geq 0$.

The case $k < 0$ is more subtle. Equation (4.31) can be rewritten using the two first of the following identities:

$$\begin{aligned}
&sinh(a+b) = sinh(a)cosh(b) + sinh(b)cosh(a)\\
&cosh(arsinh(x)) = \sqrt{1+x^2}\\
&cosh(artanh(x)) = \tfrac{1}{\sqrt{1-x^2}}\\
&sinh(artanh(x)) = \tfrac{x}{\sqrt{1-x^2}}
\end{aligned} \tag{4.32}$$

This gives a more transparent expression for R^2:

$$\boxed{R^2 = (R_i^2 - \frac{3F}{2\Lambda})cosh(\pm\sqrt{\frac{4\Lambda}{3}}(t-t_i)) + \sqrt{R_i^4 - \frac{3F}{\Lambda}R_i^2 + \frac{3f}{\Lambda}}sinh(\pm\sqrt{\frac{4\Lambda}{3}}(t-t_i)) + \frac{3F}{2\Lambda}} \tag{4.33}$$

or expressed in terms of v, k and $\omega \equiv \pm\sqrt{\frac{4\Lambda}{3}}$,

$$v = v_i cosh(\omega(t-t_i)) + \sqrt{v_i^2 + k} sinh(\omega(t-t_i)). \tag{4.34}$$

It is apparent that the three cases $k < 0$, $k = 0$ and $k > 0$ corresponds to R^2 evolving as a constant with respect to time plus $cosh(\omega(t-t_i))$, $e^{\omega(t-t_i)}$ and $sinh(\omega(t-t_i))$ respectively.

If $k < 0$, the cosh term in (4.34) will dominate and the minimal value for v^2 will be k, so that one can eliminate the sinh term by choosing the t_i as the value which minimizes v. For $k = 0$, the cosh term and the sinh term will add up to an exponential function and for $k > 0$ one can choose an initial point in time in order to eliminate the cosh term.

As seen above, R^2 will take every positive value at some point if $k > 0$ and $\Lambda > 0$. One is free to choose $R_i^2 = \frac{3F}{2\Lambda}$ for any value of r. This reduces the time dependence of (4.33) in a spherical shell at coordinate r to a hyperbolic sine only. If $k > 0$ and $\Lambda < 0$, the solution will not stay real.

If $k < 0$, then R^2 will stay real if either **a)** $\sqrt{v_i^2 + k}$ is real and $\Lambda \geq 0$ or **b)** if $\sqrt{v_i^2 + k}$ is imaginary and $\Lambda < 0$. The latter might be highly theoretical, but it may be of some academic interest.

a)

Since the spherical shell of interest evolves according to hyperbolic cosine, it is essential to find local extremal values. One way to find these is to differentiate (4.34) with respect to time. Denote the minimal/maximal value of v by v_M:

$$\dot{v}|_{v=v_M} = \omega v_i sinh(\omega(t-t_i)) + \omega\sqrt{v_i^2 + k} cosh(\omega(t-t_i)) = 0, \tag{4.35}$$

$$tanh(\omega(t-t_i)) = -\frac{\sqrt{v_i^2 + k}}{v_i}. \tag{4.36}$$

Using the two last identities from (4.32), one finds

$$v_M = v_i\sqrt{\frac{v_i^2}{-k}} - \frac{v_i^2 + k}{v_i}\sqrt{\frac{v_i^2}{-k}} = \frac{|v_i|}{v_i}\sqrt{-k} \tag{4.37}$$

or in terms of R_i^2, F, f and Λ,

$$R_M^2 = \frac{3F}{2\Lambda} \pm \sqrt{(\frac{3F}{2\Lambda})^2 - \frac{3f}{\Lambda}}. \tag{4.38}$$

This can also be found directly, as we can choose t_i and thus also R_i in a way such that $\sqrt{v_i^2 + k} = 0$ this gives an extremal value when $t = t_i$, $R^2 = R_i^2$ and this gives $v_i = \pm\sqrt{-k}$

The ultimate destiny of a solution is characterised by whether $R_i^2 \leq \frac{3F}{2\Lambda} - \sqrt{\frac{9F^2}{4\Lambda^2} - \frac{3f}{\Lambda}}$ or $R_i^2 \geq \frac{3F}{2\Lambda} + \sqrt{\frac{9F^2}{4\Lambda^2} - \frac{3f}{\Lambda}}$. The first will start out at $R^2 = 0$, then increase until it reaches the maximal value, then decrease and reach $R^2 = 0$ again in finite time. The latter case will lead to solution where R^2 "starts out at infinity" (at $t = -\infty$ if one is allowed to say so) then shrinks until it reaches a minimal value and finally bounces back out, approaching ∞ as $t \to \infty$.

It is clear that for

$$\frac{3F}{2\Lambda} - \sqrt{(\frac{3F}{2\Lambda})^2 - \frac{3f}{\Lambda}} < R_i^2 < \frac{3F}{2\Lambda} + \sqrt{(\frac{3F}{2\Lambda})^2 - \frac{3f}{\Lambda}},\ \Lambda > 0 \tag{4.39}$$

R^2 will become complex. This means that (4.39) represents an area that is either unphysical, or needs to be interpreted in a different way. As long as one has no interpretation of this mathematical phenomenon, one might just as well say that the area is forbidden. The different solutions are illustrated in Figure 4.1.

b)

Now concerning negative Λ:

The solution (4.33) can be rewritten as

$$R^2 = (R_i^2 - \frac{3F}{2\Lambda})cos(\pm\sqrt{\frac{-4\Lambda}{3}}(t-t_i)) + \sqrt{-R_i^4 + \frac{3F}{\Lambda}R_i^2 - \frac{3f}{\Lambda}}sin(\pm\sqrt{\frac{-4\Lambda}{3}}(t-t_i)) + \frac{3F}{2\Lambda}. \tag{4.40}$$

The solution for R^2 will only stay real if $R_i^4 - \frac{3F}{\lambda}R_i^2 + \frac{3f}{\lambda} = v_i^2 + k < 0$. It is seen that the solution describes oscillations with amplitude $\sqrt{-k}$ and mean value $\frac{3F}{2\lambda}$.

We are presently only considering solutions where R^2 is non-negative so it will be of interest to know whether the mean value is negative or not.
As $\Lambda < 0$, a positive mean value $\frac{3F}{2\Lambda}$ means that $F < 0$ i.e. negative curvature. Flat space, with $F = 0$ has oscillations about $R^2 = 0$ and positively curved space has a

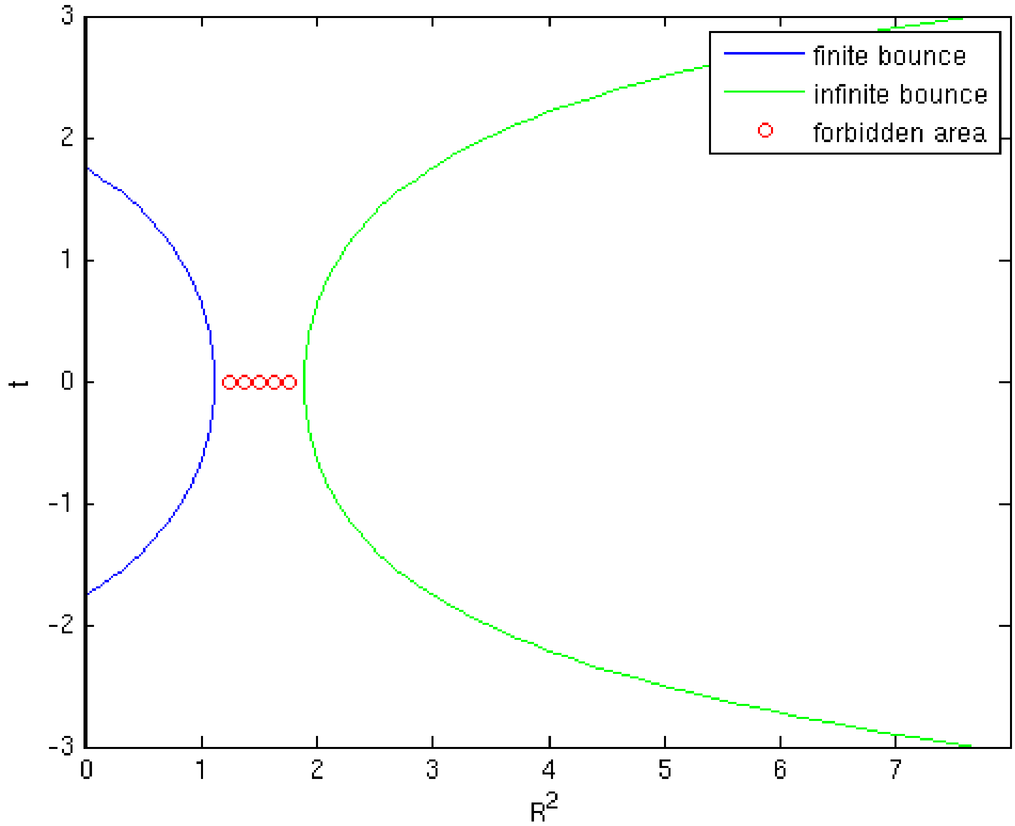


Figure 4.1: $R^2(t)$ when $\Lambda > 0$ and $k < 0$.

negative equilibrium value for $R^2 < 0$.

For negative F, if the amplitude is less than the mean value, R^2 will oscillate for ever as illustrated in Figure (4.2)

4.3 Reduction to FRW

So far, the direct consequences of equation (4.14) have been neglected. This will be repaired in this section.

Equation (4.13) is restated as

$$(\frac{M}{R^3})' = 0. \tag{4.41}$$

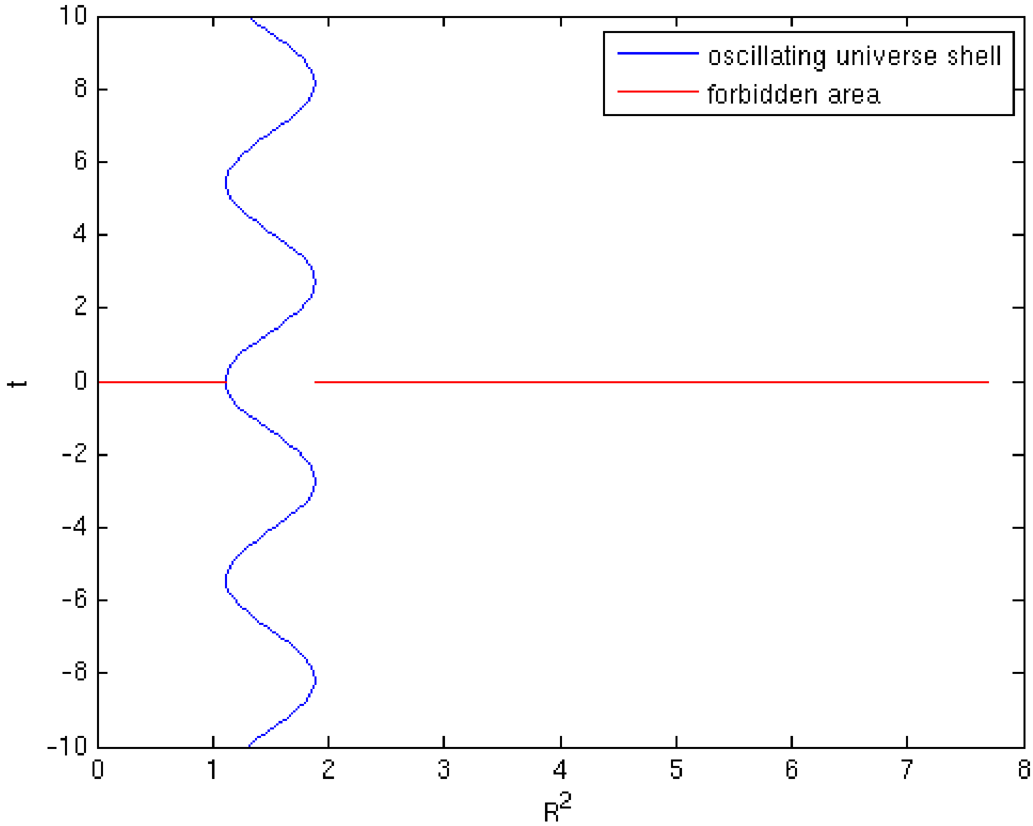


Figure 4.2: **Oscillating universe shell** The figure shows the solutions for R^2 when $\Lambda < 0$ and $0 < k < \frac{3F}{2\Lambda}$.

One can utilize this directly as

$$(\frac{M}{R^3})' = \frac{M'}{R^3} - 3M\frac{R'}{R^4} = \frac{\rho R'}{R} - \frac{3M}{R^3}\frac{R'}{R} = 0. \tag{4.42}$$

The usual assumptions of $R' \neq 0$ gives

$$\rho = \frac{3M}{R^3}, \tag{4.43}$$

but this is seen to be time independent from (4.41) and hence

$$\rho = \rho(t), \tag{4.44}$$

which means that the energy density is homogeneous.

Also, (4.43) can bee combined with $f(r) = MR$ to yield

$$R(t,r) = (\frac{3f(r)}{\rho(t)})^{\frac{1}{4}}. \tag{4.45}$$

This shows that R *factorizes.*

One can rewrite this slightly as

$$R(t,r) = \frac{a(t)}{a_i} R_i(r), \tag{4.46}$$

where $a(t) = \sqrt[4]{\frac{3}{\rho(t)}}$ and $R_i(r) = \sqrt[4]{f(r)}a_i$.

The generally inhomogeneous solution (4.33) for $\Lambda \neq 0$ was

$$R^2 = (R_i^2 - \frac{3F}{2\Lambda})cosh(\pm\sqrt{\frac{4\Lambda}{3}}(t-t_i)) + \sqrt{R_i^4 - \frac{3F}{\Lambda}R_i^2 + \frac{3f}{\Lambda}}sinh(\pm\sqrt{\frac{4\Lambda}{3}}(t-t_i)) + \frac{3F}{2\Lambda}. \tag{4.47}$$

Dividing by $\frac{R_i^2}{a_i^2}$ gives the scale factor squared,

$$a^2(t) = (a_i^2 - \frac{3F}{2\Lambda\sqrt{f}})cosh(\pm\sqrt{\frac{4\Lambda}{3}}(t-t_i)) + \sqrt{a_i^4 - \frac{3F}{\Lambda\sqrt{f}}a_i^2 + \frac{3}{\Lambda}}sinh(\pm\sqrt{\frac{4\Lambda}{3}}(t-t_i)) + \frac{3F}{2\Lambda\sqrt{f}}. \tag{4.48}$$

This leads to

$$\frac{F}{\sqrt{f}} = \Gamma \tag{4.49}$$

for some constant Γ.

The same result is obtained if $\Lambda = 0$, the solution for R^2 (4.24) is restated here,

$$R^2 = R_i^2 - F(t-t_i)^2 \mp 2\sqrt{f - FR_i^2}(t-t_i). \tag{4.50}$$

It gives

$$a^2(t) = a_i^2 - \frac{F}{\sqrt{f}}(t-t_i)^2 \mp 2\sqrt{1 - \frac{F}{\sqrt{f}}}(t-t_i). \tag{4.51}$$

The radial independence of ρ makes it simple to calculate f.

$$f = R\rho\int_0^r R^2 R' dr' = \frac{\rho}{3}R^4 \tag{4.52}$$

This gives

$$F = \Gamma\sqrt{\frac{\rho}{3}\frac{a^2}{a_i^2}}R_i^2. \tag{4.53}$$

The time independence of F implies that

$$\rho(t) = \rho_i(\frac{a(t)}{a_i})^4 \tag{4.54}$$

and

$$F = \Gamma\sqrt{\frac{\rho_i}{3}}R_i^2 \equiv kR_i^2. \tag{4.55}$$

The LTB metric (3.41) is now reduced to

$$ds^2 = -dt^2 + \frac{a^(t)}{a_i^2}(\frac{R_i'^2}{1-kR_i^2}dr^2 + R_i^2 d\theta^2 + R_i^2 sin^\theta d\phi^2). \tag{4.56}$$

This is just the FRW metric.

This concludes this chapter. It has been shown that any LTB universe with radiation and cosmological constant as the only sources can also be described completely by the FRW metric.

Chapter 5

Radiation-matter-LIVE mixture

(with consistent thermodynamics)

This chapter follows closely the work of Roberto A. Sussmann and Diego Pavón [12, 13], but also takes into account a cosmological constant. A slightly newer and more complete version of the same material is also available [11]. This version version was unknown to me at the time when i wrote this chapter and is therefore not referred to in the text. It is much more thorough and has fewer errors than the other two.

The first two sections of this chapter will be more or less identical to the material presented in these articles, with the small correction of bringing a non-zero cosmological constant along the way.

The subsequent sections contains independent analysis and I have not been able to find something similar elsewhere. The results are new as far as I am aware of.

The scope of this chapter is still the radiation dominated era, but both radiation and dust is taken into account, as well as an interaction between the two.

In order to keep difficulties on a manageable level, the following simplifications are made [12]:

a) The matter source is a fluid with shear viscosity but with neither heat conduction nor bulk viscosity.

b) The equilibrium state variables satisfy the equation of state of a mixture of relativistic and non-relativistic ideal gases, where the internal energy and the pressure of the latter have been neglected.

c) The particle numbers of each mixture component are independently conserved.

d) Dark matter and/or exotic particles are excluded, but a tight coupling between radiation and matter keeps the system in *Local Thermal Equilibrium* (LTE).

The justification of these simplifications are also to be found in [12]:

The condition a) is not very generally valid and shrinks the range of validity of this model significantly. Shear viscosity will dominate over heat conduction in the limit of high temperatures. The bulk viscosity is negligible in the temperature range $10^3K < T < 10^6K$. The restriction implies an adiabatic (no heat flow) but irreversible (non-zero viscosity) evolution of the system.

The neglecting of everything except the rest mass of the dust particles in restriction b) is due to the high ratio of photons to baryons in the radiation dominated era $\approx 10^9$.

After nucleosynthesis, with $T < 10^6K$, matter creation/ annihilation processes balance each other out and cease to be dynamically important.

The condition d) gives a more a priori restriction on the model as dark matter is outside the scope of the present investigation.

5.1 The Energy-Momentum Tensor

The matter- and radiation content of the universe will be modeled as one ultrarelativistic gas of massless particles and a non-relativistic, monatomic ideal gas.

The particle densities and temperatures of the dust and radiation will be denoted by $n^{(m)}$, $n^{(r)}$ and $T^{(m)}$, $T^{(r)}$ respectively as in [12].

The energy density can then be written as

$$\rho = mn^{(m)} + \frac{3}{2}n^{(m)}k_BT^{(m)} + 3n^{(r)}k_BT^{(r)}, \tag{5.1}$$

while the pressure takes the form

$$p = n^{(m)}k_BT^{(m)} + n^{(r)}k_BT^{(m)}. \tag{5.2}$$

The temperatures of the two gases will be equal as long as the the system stays under LTE. As LTE will be assumed, $T^{(m)} = T^{(r)} \equiv T$.

Imposing also the other restrictions listed above gives simplified expressions for the energy density and pressure

$$\rho = mn^{(m)} + 3n^{(r)}k_BT \equiv \rho^{(m)} + \rho^{(r)} \tag{5.3}$$

and

$$p = n^{(r)}k_BT \equiv \frac{1}{3}\rho^{(r)}. \tag{5.4}$$

The energy momentum tensor takes the form

$$T^{\mu\nu} = \rho u^\mu u^\nu + ph^{\mu\nu} + \Pi^{\mu\nu} \tag{5.5}$$

where $h^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$ is the tensor of projection onto the plane of simultaneity, that is, perpendicular to the four-velocity. $\Pi^{\mu\nu}$ is the *shear viscous pressure tensor* due to the particle-radiation interaction. It is symmetric, trace free and orthogonal to the four-velocity field, i.e. $u_\mu \Pi^{\mu\nu} = 0$.

It may seem strange that it is assumed that the four velocity of radiation is equal to that of dust, as the four-velocity of a photon is a light-like vector, while the four-velocity of a massive particle is time-like. One should keep in mind that the four-velocity is interpreted as averaged over a large area, as pointed out in subsection 3.2.2.

The conservation of particle number densities can be expressed as

$$(n^{(m)} u^\mu)_{;\mu} = (n^{(r)} u^\mu)_{;\mu} = 0 \tag{5.6}$$

Calculating the covariant derivative, using the Christoffel symbols (3.53) gives

$$(nu^\mu)_{;\mu} = \dot{n} + n\frac{\dot{R}'}{R'} + 2n\frac{\dot{R}}{R} = \frac{\frac{d}{dt}(nR'R^2)}{R^2 R'} = 0. \tag{5.7}$$

As long as $R^2 R' \neq 0$ as usually, this means that

$$n^{(m)} = n_i^{(m)} (\frac{R_i}{R})^2 \frac{R_i'}{R'}, \; n^{(r)} = n_i^{(r)} (\frac{R_i}{R})^2 \frac{R_i'}{R'}. \tag{5.8}$$

The most general form of the interaction term $\Pi^{\mu\nu}$ can be deduced from the property that it is orthogonal to the four velocity, it is diagonal due to comoving coordinates and its θ and ϕ components must be equal due to spherical symmetry. It has the same form as the shear tensor, (3.56).

$$\Pi^\mu_\nu = diag(0, \; -2P(r,t) \; , P(r,t), \; P(r,t)) \tag{5.9}$$

5.2 Field equations

The field equations are arranged in the same way as was done in the beginning of section 4.1, except (5.11) where each side of the equality sign is equal to $T^{\hat{r}\hat{r}} + T^{\hat{\theta}\hat{\theta}} + T^{\hat{\phi}\hat{\phi}}$. They are also found in [12], although with a sign error in the first equation.

$$\frac{[R(\dot{R}^2+F)-\frac{\Lambda}{3}R^3]'}{R^2R'}=\rho \tag{5.10}$$

$$-\frac{[R(\dot{R}^2+F)+2\ddot{R}R^2-\Lambda R^3]'}{3R^2R'}=p \tag{5.11}$$

$$\frac{R}{6R'}[\frac{R(\dot{R}^2+F)+2R^2\ddot{R}}{R^3}]'=P \tag{5.12}$$

From the conservation of particle density and the constancy of the rest masses of dust particles, it follows that the total energy due to dust must be constant. In particular,

$$M(r)\equiv\int_0^r \rho^{(m)}R^2R'dr'=\int_0^r \rho_i^{(m)}R_i^2R_i'dr'. \tag{5.13}$$

Also, define $\bar{W}$ as

$$\bar{W}=R\int_0^r \rho^{(r)}R^2R'dr'. \tag{5.14}$$

The Einstein equations (5.10) and (5.11) can be integrated with respect to r to yield

$$R(\dot{R}^2+F)-\frac{\Lambda}{3}R^3=M+\frac{1}{R}\bar{W} \tag{5.15}$$

and

$$-R(\dot{R}^2+F)-2\ddot{R}R^2+\Lambda R^3=\frac{1}{R}\bar{W}. \tag{5.16}$$

Combining these gives

$$2R(\dot{R}^2+F)+2\ddot{R}R^2-\frac{4}{3}\Lambda R^3=M. \tag{5.17}$$

From (5.16), the time derivative of $\bar{W}$ is found to be

$$\dot{\bar{W}}=\dot{R}(2R(\dot{R}^2+F)+2\ddot{R}R^2-M+\frac{4\Lambda}{3}R^3). \tag{5.18}$$

Utilizing (5.17), this gives

$$\dot{\bar{W}} = 0, \tag{5.19}$$

or

$$\bar{W} = \bar{W}_i. \tag{5.20}$$

Inserting this into equation (5.15) gives

$$\dot{R}^2 = \frac{M}{R} + \frac{\bar{W}_i}{R^2} - F + \frac{\Lambda}{3}R^2 \tag{5.21}$$

or

$$\boxed{\dot{R}^2 = \frac{M}{R} + W\frac{R_i}{R^2} - F + \frac{\Lambda}{3}R^2,} \tag{5.22}$$

where the function W is defined as

$$W(r) = \int_0^r \rho_i^{(r)} R_i^2 R_i' dr'. \tag{5.23}$$

From here, the article [12] proceeds by setting $F = 0$, thus neglecting curvature. I will deviate from this approach and first consider only the simplification $\Lambda = 0$. Equation (5.22) will be analytically integrable, however, it can not be analytically solved for R.

5.3 Non-zero curvature with $\Lambda = 0$

The cosmological constant makes the equation (5.22) difficult to integrate, so I will focus on the case $\Lambda = 0$, but let the curvature be non-zero. This gives the following starting point:

$$\boxed{R^2\dot{R}^2 = -FR^2 + MR + WR_i} \tag{5.24}$$

5.3.1 Integration

Introducing $\omega = \frac{-WR_i}{F}$, $\mu = -\frac{M}{F}$, this can be rewritten as

$$\frac{R + \frac{\mu}{2} - \frac{\mu}{2}}{\sqrt{R^2 + \mu R + \omega}} dR = \pm\sqrt{-F}dt. \tag{5.25}$$

Splitting the fraction in two terms and doing a change of variables gives

$$\frac{1}{2}\frac{1}{\sqrt{(R+\frac{\mu}{2})^2+\omega-\frac{\mu^2}{4}}}d((R+\frac{\mu}{2})^2)-\frac{\mu}{2}\frac{1}{\sqrt{(R+\frac{\mu}{2})^2+\omega-\frac{\mu^2}{4}}}dR=\pm\sqrt{-F}dt. \quad (5.26)$$

Integrating from t_i to t gives the final result in (5.27).

$$\boxed{\sqrt{R^2+R\mu+\omega}-\sqrt{R_i^2+R_i\mu+\omega}-\frac{\mu}{2}ln(\frac{\sqrt{R^2+\mu R+\omega}+R+\frac{\mu}{2}}{\sqrt{R_i^2+\mu R_i+\omega}+R_i+\frac{\mu}{2}})=\pm\sqrt{-F}(t-t_i)} \quad (5.27)$$

Contrary to the case where $F=0$, the above equation is transcendental and it can not be solved for R. It is possible though to plot t as a function of R and reverse the axes.

The exact solution for R is not used in [12], it might be possible to do something similar even without the analytic solution. I am not going to pursuit this approach much further, but at least rewrite equation (5.27) so that it gets a similar form to the one used in the article.

Introducing new variables, thereby also redefining μ,

$$\mu=\frac{M}{FR_i}, \quad \epsilon=\frac{W}{M}, \quad y=\frac{R}{R_i} \text{ and } k_i=\frac{F}{R_i^2} \quad (5.28)$$

transforms (5.27) into

$$\sqrt{y^2-\mu(y+\epsilon)}-\sqrt{1-\mu(1+\epsilon)}+\frac{\mu}{2}ln(\frac{\sqrt{y^2-\mu(y+\epsilon)}+y-\frac{\mu}{2}}{\sqrt{1-\mu(1+\epsilon)}+1-\frac{\mu}{2}})=\pm\sqrt{-k_i}(t-t_i). \quad (5.29)$$

This is similar to the starting point of the bulk of the analysis in the article. However, continuing the procedure with this more general expression seems to be much more difficult and I will not do this here.
Instead, I will analyse the different solutions of (5.24) and solve (5.27) for R numerically.

5.3.2 Solution analysis

The solutions of (5.24) will be analysed and plotted, but first, I will discuss the different types of solutions which may occur.

I emphasize that I consider the time evolution at a specific r-coordinate, say r_0. In the following, I frequently write $R(t)$ for $R(t, r_0)$ (and R for $R(t)$).

An important property of a solution is whether it is monotone or not. If $\dot{R} \neq 0$ for all t, R will be monotone. It may take values on the whole of $\mathbb{R}$ or it may converge to some value for $t \to \infty$. If there is a point in time when $\dot{R} = 0$, $\dot{R}$ may switch sign at this point.

In the case of non-zero R, the condition $\dot{R} = 0$ is formulated as

$$-FR^2 + MR + WR_i = 0. \tag{5.30}$$

This is readily solved for R, giving

$$R_{\pm} = \frac{M \pm \sqrt{M^2 + 4FWR_i}}{2F}. \tag{5.31}$$

To check if $\dot{R}$ really changes sign when $R = R_{\pm}$, one should investigate how $\ddot{R}$ behaves.

Dividing (5.24) by R^2 and differentiating with respect to time gives

$$2\dot{R}\ddot{R} = -\frac{M\dot{R}}{R^2} - 2\frac{WR_i\dot{R}}{R^3}. \tag{5.32}$$

For $\dot{R} \neq 0$, this gives

$$\ddot{R} = -\frac{M}{2R^2} - \frac{WR_i}{R^3} \tag{5.33}$$

and the condition that this is 0 is written as

$$R = -\frac{2WR_i}{M}. \tag{5.34}$$

If one assumes that $\ddot{R}$ is continuous, condition (5.34) is necessary for $\ddot{R} = 0$ even if $\dot{R} = 0$. (Also, condition (5.34) can be deduced from adding together (5.10) and (5.11).) What to draw from this is that if $\dot{R} = 0$, then $\ddot{R} \neq 0$, except if also (5.34) holds.

in this case,

$$R_{\pm} = \frac{M \pm \sqrt{M^2 + 4FWR_i}}{2F} = -\frac{2WR_i}{M}, \tag{5.35}$$

hence

$$M^2(M^2 + 4FWR_i) = M^4 + 8M^2FWR_i + 16(FWR_i)^2 \tag{5.36}$$

and

$$M^2 = -4FWR_i. \tag{5.37}$$

The exception is apparently the case when $R_+ = R_- = \frac{M}{2F}$. Also, one should note that in this case, inserting $R = R_{\pm}$ in equation (5.27) leads to $t = \infty$, this means that it is not necessary to check what $\ddot{R}$ is when $R = R_\pm$, because R only converges to $R_\pm$ when $t \to \infty$.

An obvious constraint on the solutions is that $\dot{R}^2 \geq 0$. This means that for $R_\pm \in \mathbb{R}$, if $F < 0$, either $R \leq R_-$ or $R \geq R_+$ and if $F > 0$, then $R_- \leq R < R_+$. For $R_\pm \notin \mathbb{R}$, F must be negative.

Finally, before plotting, observe that if R goes to zero and $R \neq R_\pm$, then $\dot{R}$ diverges. Also, one should keep in mind that when plotting t as a function of R and reversing the axes, one does not necessarily get R as a function of t, as $t(R)$ does not need to be injective. In order to get $R(t)$, segments which have endpoints where R as well as its derivatives coincide must be patched together. This subtlety makes sense with R not being analytic.

5.3.3 Plotted solutions

In this subsection, the different solutions I considered in the previous subsection are plotted by plotting t as a function of R and reversing the axes.

Throughout this subsection, red curves correspond to the minus sign in (5.27) and blue curves correspond to plus.

Complex roots First consider the cases where $R_\pm \notin \mathbb{R}$, implying that $\dot{R} \neq 0$ for every t. The solutions are plotted in figure 5.1.

As the point where the functions have been glued together are at $R = 0$, the shifting of the functions is really unnecessary because at $R = 0$, the theory would collapse in any case.

Note that negative R is no problem, as it only enters squared in squared terms in the metric. One should keep in mind, though, that if R is negative, a positive M or W means that the corresponding energy density is negative.

Bouncing solutions The next cases to be considered are the cases when $F < 0$ and $R_+ \neq R_-$. The curves will attach perfectly as long as $R_- \leq R_i \leq R_+$ is chosen.

The curves are going to look quite similar to the hyperbolic cosine solutions from chapter 4, but there will be some differences. Figure 5.2 shows the time dependence when $R_- < 0 < R_+$.

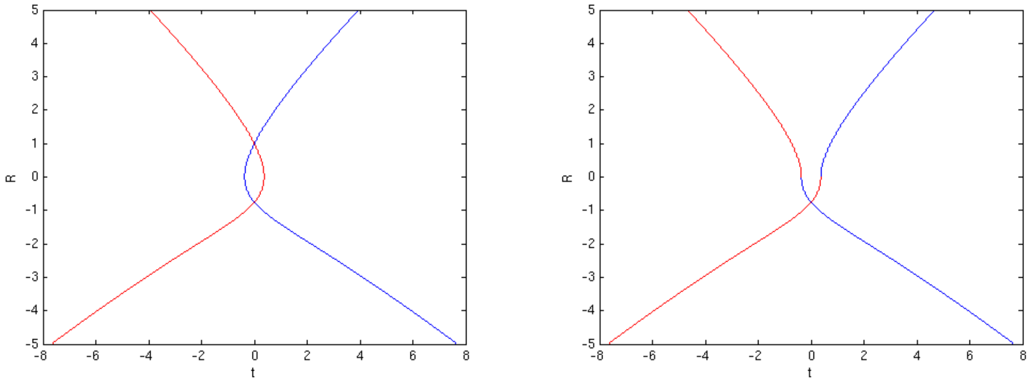


Figure 5.1: **Time evolution with complex roots** $M = W = R_i = 1$, $F = -\frac{1}{2}$ In the right figure, the two solutions have been shifted in order to be functions of t.

It is also interesting to see what kind of solutions occur if for example both $R_\pm > 0$. The dependence which appears is plotted in figure 5.3.

Asymptotically static cases This paragraph deals with cases where the radiation and dust have opposite signs. In most models describing our universe, both radiation and dust have positive energy densities, but if the energy density of one of them is allowed to be negative, the solutions described here may occur.

These situations are identified with $R_+ = R_-$ and $F < 0$. If $R_\pm, R_i > 0$, this means that M is positive and W is negative. Equivalently, if $R_\pm, R_i < 0$, M is negative and W is positive. The nice invariance is that in both cases, $\rho^{(r)} < 0$ and $\rho^{(m)} > 0$. Either way, the radiation and dust will balance each other out perfectly at $R_\pm$. Assuming $R_\pm > 0$, $R_i < R_\pm$ will imply expansion, as the negative energy density of radiation dominates that of dust, and for $R_i > R_\pm$ the universe will contract. In both cases, $R \to R_\pm$.

One may of course also reverse the time direction, turning things the other way around. The solutions are plotted in figure 5.4.

Oscillating solutions The final case to consider is the case with $F > 0$ and $R_- \leq R \leq R_+$. This will permit solutions which oscillate. I will first consider the case when both $R_\pm > 0$. This gives $M > 0$ and $W < 0$. There is a similar invariance as with the asymptotic cases. If $R_i, R_\pm < 0$, the energy density of radiation will still be negative, while the energy density of dust remains positive. The oscillating behaviour is seen in figure 5.5.

The case where $R_- < 0 < R_+$ is shown in figure 5.6. It gives solutions which oscillate between $R_- < 0$ and $R_+ > 0$ passing through $R = 0$ where the theory

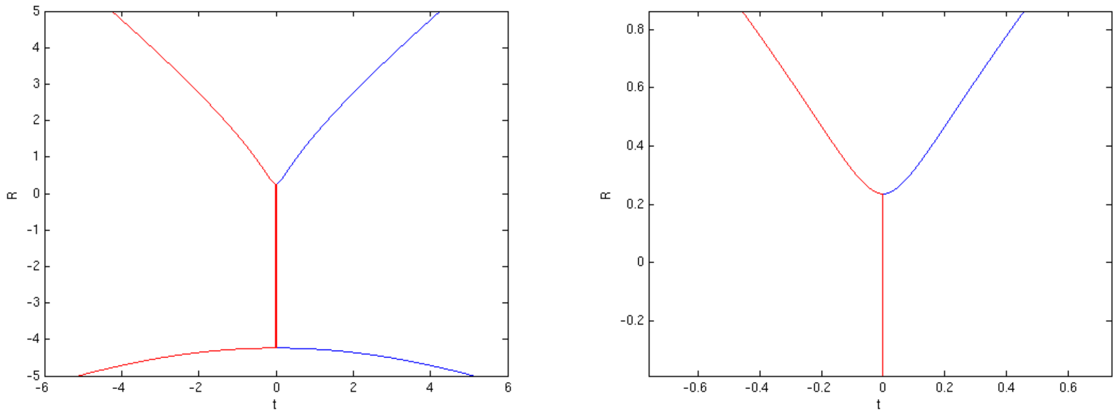


Figure 5.2: **Bouncing solutions** $M = 2$, $WR_i = F = -\frac{1}{2}$ $R_i = -2 - \sqrt{5}$, The positive and negative time solutions match perfectly. R starts at infinity and reaches $R_{\pm}$ before bouncing back. Note that if the turning point is close to $R = 0$, the bounce may seem abrupt. However, zooming in, as seen in the right figure, shows that the bounce is smooth.

breaks down. For example, the pressure of dust will no longer be approximately zero.

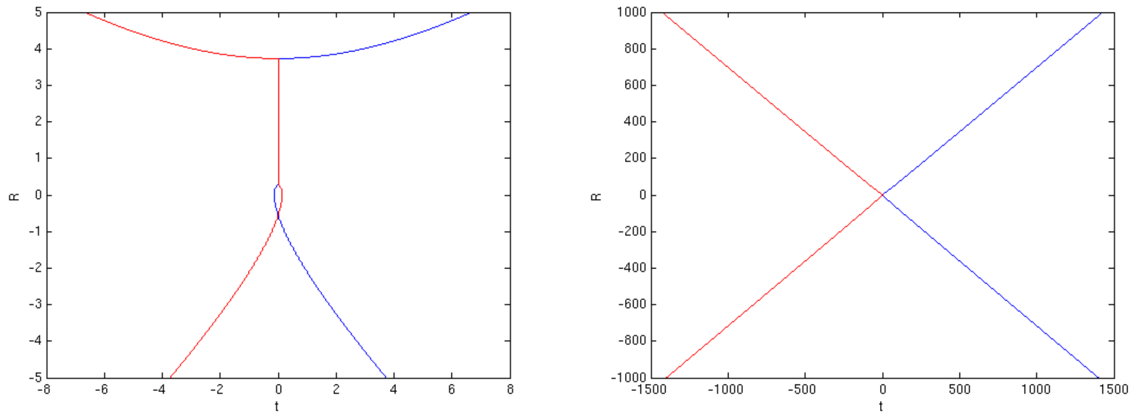


Figure 5.3: **Bouncing through R=0** $M = -2$, $WR_i = \frac{1}{2}$, $F = -\frac{1}{2}$ $R_i = 2 - \sqrt{3}$, Also here, the positive and negative time solutions match perfectly. The topmost solution is similar to the ones in figure 5.2. The down-most solution must be seen as three different cases. One solution is that the universe is collapsing, starting at $t = -\infty$ and ending at a finite point in time. A similar solution starts at some finite time and expands for ever. The small part between $R = 0$ and $R = R_-$ will be a universe with a finite lifetime. It is interesting to see that the expanding solution with $R < 0$ seems to decelerate, while the solution with $R > 0$ experiences acceleration. Zooming out shows that they behave asymptotically similarly, though.

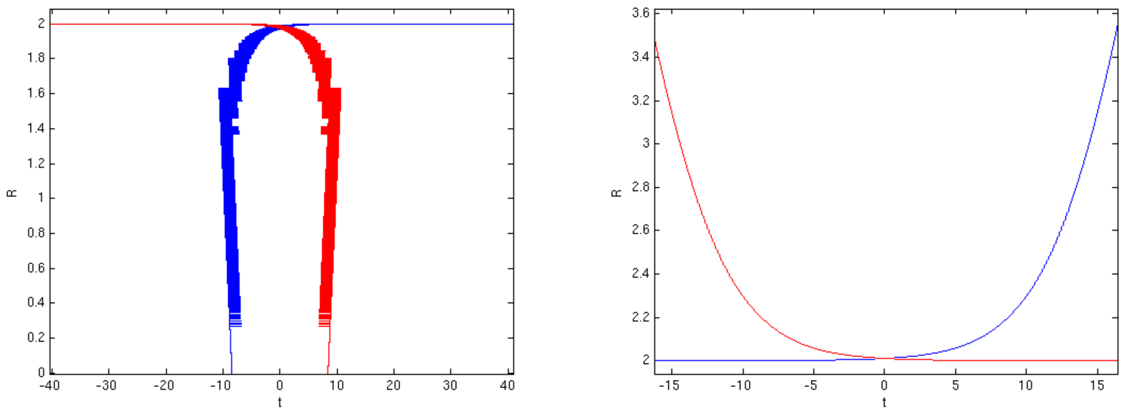


Figure 5.4: **Asymptotically static cases** $M = 2$, $WR_i = -2$, $F = -\frac{1}{2}$ and $R_i = 1.99$ in the left figure, $R_i = 2.01$ in the right figure. It seems that numerical errors with complex numbers cause some irregularities in the left figure, and one should not put too much faith in it. The dependence is seen though, R will approach $R_\pm$ at either ∞ or $-\infty$

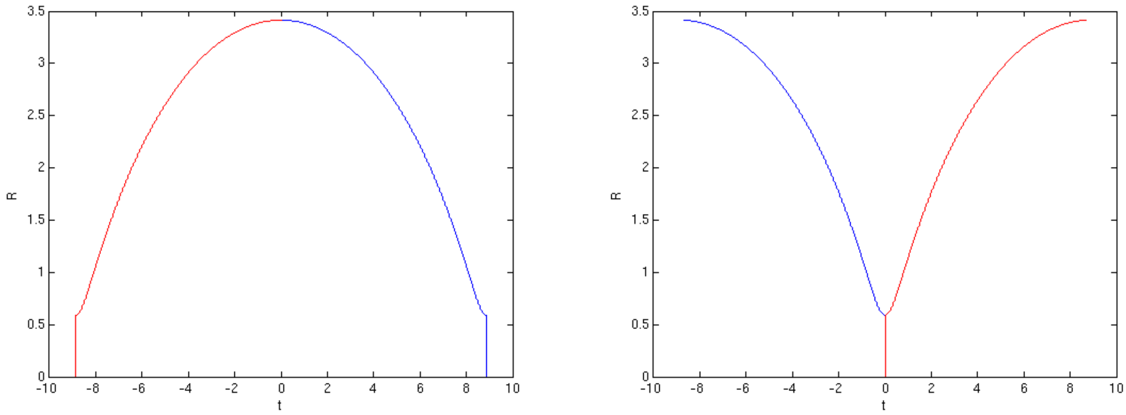


Figure 5.5: **Oscillations with $R > 0$** $M = 2,\ WR_i = -1,\ F = \frac{1}{2}$ and $R_i = R_+ = 2 + \sqrt{2}$ in the left figure, $R_i = R_- = 2 - \sqrt{2}$ in the right figure. Both figures shows one period of oscillation. The concave time intervals correspond to dust domination, decelerating the expansion and starting contraction, while the much shorter convex time intervals correspond to radiation dominated periods.

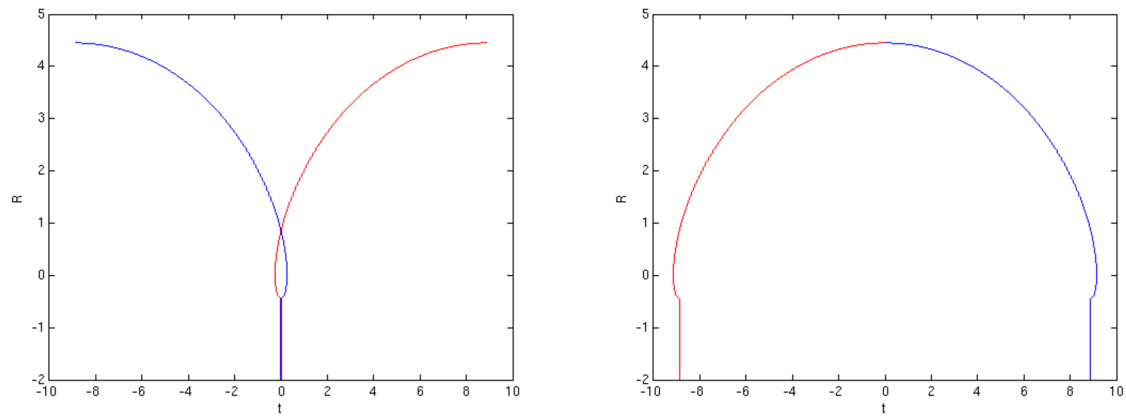


Figure 5.6: **Oscillations through $R = 0$** It is clear that one can not without explanation assume that the theory does not break down when $R = 0$. However, if one were to consider a universe which evolves completely as the equations predict, this universe would experience oscillations where R^2 oscillates with alternating amplitude. On the other hand, if $R = 0$ is forbidden, there will be two different solutions with finite life-time spanning between $R = 0$ and $R = 0$ corresponding to negative and positive R.

Chapter 6

LTB with $p = \omega\rho$

The following will be assumed:

$$ds^2 = -dt^2 + \frac{R'(t,r)^2}{1-F(r)}dr^2 + R^2(t,r)(d\theta^2 + sin^2\theta d\phi^2) \tag{6.1}$$

$$T^{\mu\nu} = \rho(t,r) \cdot diag(1,\ \omega(t) + 2\alpha(t),\ \omega(t) - \alpha(t),\ \omega(t) - \alpha(t)) \tag{6.2}$$

For some functions of time ω and α.

That is, the LTB as well as a source with equation of state $p = \omega\rho$ and some anisotropic pressure proportional to ρ. The motivation behind this anisotropic pressure is just that it doesn't make the Einstein equations much more difficult to solve. One might just as well include it to obtain a more general result. Also, it turns out that allowing ω and α to vary in time causes no difficulties either.

The Einstein equations take the following form when assuming $R \neq 0 \neq R'$:

$$\frac{2\dot{R}\dot{R}'R + r'(\dot{R}^2 + F) + F'R}{R'R^2} = \Lambda + \rho \tag{6.3}$$

$$-\frac{2\ddot{R}R + \dot{R}^2 + F}{R^2} = -\Lambda + (\omega + 2\alpha)\rho \tag{6.4}$$

$$-\frac{\frac{1}{2}F'R + \ddot{R}'R^2 + \ddot{R}R'R + \dot{R}\dot{R}'R}{R'R^2} = -\Lambda + (\omega - \alpha)\rho \tag{6.5}$$

The integrated equations take the form

$$R(\dot{R}^2 + F) - \frac{\Lambda}{3}R^3 = M \tag{6.6}$$

$$2R(\dot{R}^2 + F) + 2\ddot{R}R^2 = \frac{4\Lambda}{3}R^3 + (1 - 3\omega)M \tag{6.7}$$

$$2\ddot{R}R^2 + R(\dot{R}^2 + F) = (\beta(t) + 6\alpha\bar{M})R^3 \tag{6.8}$$

Here, $M = \int_0^r \rho R' R^2 dr'$, $\beta = \lim_{r\to 0} \frac{2\ddot{R}R + R(\dot{R}+F)}{R^3}$, and $\bar{M} = \int_0^r dr' \rho \frac{R'}{R}$.

Subtracting the second equation from the sum of the other two gives

$$(\beta + 6\alpha\bar{M} - \Lambda)R^3 + 3\omega M = 0, \tag{6.9}$$

or

$$\beta + 6\alpha\bar{M} - \Lambda + \frac{3\omega M}{R^3} = 0. \tag{6.10}$$

Differentiation with respect to r gives

$$(2\alpha + \omega)R^3\rho = 3\omega M. \tag{6.11}$$

Differentiating again shows that

$$(6\alpha + 3\omega)(\rho' R^3 + 3\rho R^2 R') = 9\omega\rho R' R^2, \tag{6.12}$$

$$(2\alpha + \omega)\rho' + 6\alpha\rho\frac{R'}{R} = 0. \tag{6.13}$$

It is clear that $2\alpha + \omega = 0$ implies $\rho = 0$ unless $\omega = \alpha = 0$. In the following $\boxed{\omega + 2\alpha \neq 0}$ will be assumed.

Rewriting (6.13) slightly gives

$$R^{\frac{6\alpha}{2\alpha+\omega}}\rho' + \frac{6\alpha}{2\alpha+\omega}R^{\frac{6\alpha}{2\alpha+\omega}-1}\rho R' = (R^{\frac{6\alpha}{2\alpha+\omega}}\rho)' = 0. \tag{6.14}$$

This gives for M that

$$M = \int \rho R^2 R' dr = \rho R^{\frac{6\alpha}{2\alpha+\omega}} \int R^{2-\frac{6\alpha}{2\alpha+\omega}} R' dr = \frac{2\alpha+\omega}{3\omega}\rho R^3 \tag{6.15}$$

and

$$(MR^{\frac{-3\omega}{2\alpha+\omega}})' = 0. \tag{6.16}$$

This may be written as

$$\frac{M'}{M} = \frac{3\omega}{2\alpha+\omega}\frac{R'}{R} \tag{6.17}$$

The assumption $\boxed{\omega \neq 0}$ will be added in the following.

A similar result as (4.10) is obtained when taking the time derivative of (6.6) and dividing by M'.

$$\frac{\dot{M}}{M'} = -(\omega + 2\alpha)\frac{\dot{R}}{R'} \tag{6.18}$$

Equations (6.17) and (6.18) gives

$$\frac{\dot{M}}{M} = -3\omega\frac{\dot{R}}{R}. \tag{6.19}$$

In terms of ρ, this says

$$\frac{\dot{\rho}}{\rho} - \frac{\dot{\omega}}{\omega} + \frac{\dot{\omega}+2\dot{\alpha}}{\omega+2\alpha} = -3(\omega+1)\frac{\dot{R}}{R} \tag{6.20}$$

or

$$\frac{\partial}{\partial t}(\frac{2\alpha+\omega}{\omega}\rho R^{3(\omega+1)}) = 3(\frac{2\alpha+\omega}{\omega}\rho R^{3(\omega+1)})\dot{\omega}ln(R). \tag{6.21}$$

It proves useful to introduce

$$\boxed{q(t) \equiv \rho R^{\frac{6\alpha}{2\alpha+\omega}}.} \tag{6.22}$$

The radial independence can be seen from the equations (6.15) and (6.19). Equation (6.20) can now be written

$$\frac{\dot{q}}{q} - ln(R)\frac{\partial}{\partial t}(\frac{6\alpha}{2\alpha+\omega}) - \frac{\dot{\omega}}{\omega} + \frac{\dot{\omega}+2\dot{\alpha}}{\omega+2\alpha} = (\frac{6\alpha}{2\alpha+\omega} - 3(\omega+1))\frac{\dot{R}}{R}. \tag{6.23}$$

Differentiating with respect to r gives

$$-\frac{R'}{R}\frac{\partial}{\partial t}(\frac{6\alpha}{2\alpha+\omega}) = (\frac{6\alpha}{2\alpha+\omega} - 3(\omega+1))(\frac{\dot{R}'}{R} - \frac{\dot{R}R'}{R^2}). \tag{6.24}$$

6.1 Factorization

It is helpful to know when $R(r,t)$ can be written as a product of one function depending on time and one function depending on the radial coordinate. Whenever this is the case, I will write

$$R(r,t) \equiv a(t)R_i(r). \tag{6.25}$$

By definition, $a_i = a(t_i) = 1$.

If R can be written on this form, it is implied that

$$\frac{\partial}{\partial t}(\frac{R'}{R}) = \frac{\partial}{\partial t}(\frac{R'_i}{R_i}) = 0. \tag{6.26}$$

This means that

$$\frac{\partial}{\partial t}(\frac{R'}{R}) = 0 \Leftrightarrow \frac{\dot{R}'}{R'} - \frac{\dot{R}}{R} = 0 \tag{6.27}$$

is a necessary condition for R to factorize.

For the converse, assume

$$\frac{\partial}{\partial t}(\frac{R'}{R}) = 0 \text{ or equivalently } \frac{R'}{R} = \frac{R'_i}{R_i}. \tag{6.28}$$

Rearranging,

$$R = R_i\frac{R'}{R'_i} \tag{6.29}$$

and all that has to be done is to show that $\frac{R'}{R'_i} = a(t)$ is independent of radial coordinate.

$$(\frac{R'}{R'_i})' = (\frac{R}{R_i})' = \frac{R'}{R_i} - \frac{RR'_i}{R_i^2} = \frac{R'}{R_i}(1 - \frac{RR'_i}{R_iR'}) = 0, \tag{6.30}$$

hence, R factorizes.

I could have done a similar proof with the time and radial derivatives interchanged. The final result is that

$$\boxed{R = a(t)R_i(t) \Leftrightarrow \frac{\partial}{\partial t}(\frac{R'}{R}) = 0 \Leftrightarrow \frac{\partial}{\partial r}(\frac{\dot{R}}{R}) = 0.} \tag{6.31}$$

Equation (6.24) gives good bounds on when this is the case. Rewriting it slightly gives

$$\frac{\partial}{\partial t}(\frac{6\alpha}{2\alpha+\omega}) = (\frac{6\alpha}{2\alpha+\omega} - 3(\omega+1))(\frac{\dot{R}'}{R'} - \frac{\dot{R}}{R}). \tag{6.32}$$

As shown above, R factorizes iff $\frac{\dot{R}'}{R'} - \frac{\dot{R}}{R} = 0$. From (6.32), it is obtained that

$$R = a(t)R_i(t) \implies \frac{\partial}{\partial t}(\frac{6\alpha}{2\alpha+\omega}) = 0, \tag{6.33}$$

Which means that $\frac{\partial}{\partial t}(\frac{6\alpha}{2\alpha+\omega}) = 0$ is a necessary condition for R to factorize. In the case $\omega = 0$, this is trivially fulfilled, but then $(\frac{6\alpha}{2\alpha+0} - 3(0+1)) = 0$ also, so equation (6.32) does not give anything. In the other case, $\omega \neq 0$ and one can write

$$\alpha \equiv k\omega, \tag{6.34}$$

where k is a constant.

This is a necessary condition for R to factorize, but not necessarily a sufficient one. For the rest of this section, (6.34) will be assumed and further conditions which ascertains factorization of R will be derived.

The cases which may cause trouble are the ones with

$$\frac{6\alpha}{2\alpha+\omega} - 3(\omega+1) = 0. \tag{6.35}$$

Inserting $\alpha = k\omega$ gives

$$k = \frac{1}{2}\frac{\omega+\omega^2}{1-\omega-\omega^2}, \tag{6.36}$$

which implies that ω is constant and $\omega^2+\omega = \frac{k}{1+2k}$. This means that R may or may not be factorizable on an interval where

$$\dot{\omega} = 0, \quad \alpha = \frac{-\omega}{2}\frac{\omega^2+\omega}{\omega^2+\omega-1}. \tag{6.37}$$

I will now show that this only applies if the interval includes every point of time. Suppose that there exists such an interval, then differentiation of (6.32) with respect to time leads to

$$\frac{\partial}{\partial t}(\frac{\dot{R}'}{R'} - \frac{\dot{R}}{R}) = 0 \tag{6.38}$$

on the interval. If the chosen interval is not contained in an other such interval (the chosen interval is maximal), then

$$\frac{\dot{R}'}{R'} - \frac{\dot{R}}{R} = 0 \tag{6.39}$$

on the boundary. The interval has a boundary iff it is not the whole range of time. In the case of a boundary (6.38) implies that (6.39) is valid on the whole interval.

The final result is that

$$\boxed{\alpha = k\omega,\ 2\alpha + \omega \neq 0 \neq \omega,\ \omega + \omega^2 \neq \frac{k}{1+2k} \forall t \implies R = a(t)R_i(t) \qquad (6.40)}$$

6.2 The homogeneous cases with $\alpha = 0,\ \omega \neq 0$

In the special case $\alpha = 0,\ \omega \neq 0$, covering e.g. a radiation dominated LTB universe with a cosmological constant, it is obtained from (6.13) that

$$\rho' = 0. \tag{6.41}$$

Hence $\rho = \rho(t)$ and **the distribution of matter is homogeneous**. It remains to show that the metric reduces to a FRW metric as well.

Equation (6.6) can be rewritten, using equation (6.15) as

$$\frac{F}{R^2} = \frac{\Lambda + \rho}{3} - \frac{\dot{R}^2}{R^2}. \tag{6.42}$$

If there is some point in time t_i such that $\omega(t_i) \neq -1$, it follows from (6.20) that the right hand side of equation (6.42) is a function of time only for time intervals where $\omega \neq -1$. Evaluating at t_i gives

$$\frac{F}{R_i^2} = \frac{\Lambda + \rho_i}{3} - \frac{\dot{R_i}^2}{R_i^2} \equiv k_0 \tag{6.43}$$

and

$$F = k_0 R_i^2. \tag{6.44}$$

The case where $\omega(t) = -1$ for all t is not as restricted and it turns out that it is not necessarily FRW. The studying of this case is the subject of the subsequent section.

As the the homogeneous case of this section fulfills the conditions of the implication in (6.40) with $k = 0$, one can write

$$ds^2 = -dt^2 + a^2(t)(\frac{R_i'^2}{1 - k_0 R_i^2} + d\theta^2 + sin^2\theta d\phi^2), \tag{6.45}$$

which is indeed the Friedman-Robertson-Walker metric.

6.3 LTB de Sitter Universe

In the case $\omega = -1$, (6.20) implies that ρ is constant. This allows for writing equation (6.6) as

$$\frac{dR}{\sqrt{\frac{3F}{\Lambda+\rho} + R^2}} = \sqrt{\frac{\Lambda + \rho}{3}} dt \tag{6.46}$$

whenever $\Lambda + \rho \neq 0$ and

$$dR = \sqrt{F} dt \tag{6.47}$$

if $\Lambda + \rho = 0$.

As ρ behaves just like a cosmological constant, one might as well absorb it into Λ, so that ρ is treated as 0.

Integration of equations (6.46) and (6.47) gives

$$R = R_i cosh(\sqrt{\frac{\Lambda}{3}}(t - t_i)) + \sqrt{R_i^2 + \frac{3F}{\Lambda}} sinh(\sqrt{\frac{\Lambda}{3}}(t - t_i)) \tag{6.48}$$

and

$$R = R_i + \sqrt{F}(t - t_i). \tag{6.49}$$

The solution (6.48) is found in the same way as the solution for R^2 in chapter 4. There is a big difference though, as R in the de Sitter model behaves similarly to the R^2 of the radiation dominated model.

Having learned a lesson with the radiation dominated model, I would like to check that there are no further restrictions on the solutions.

The Einstein equations can now be written as

$$\dot{R}^2 + F = \Lambda R^2 \tag{6.50}$$

$$\dot{R}^2 + F + \ddot{R}R = \frac{2\Lambda}{3} R^2 \tag{6.51}$$

$$2\ddot{R}R + \dot{R}^2 + F = \beta(t)R^2 \tag{6.52}$$

The Einstein equations can only be fulfilled if $\beta = \Lambda$ and in this case equation (6.52) is an implication of equations (6.50) and (6.51).
The solutions (6.48) and (6.49) are derived from (6.50), so all that has to be done is to check that they satisfy (6.51) as well. As (6.50) is fulfilled, this amounts to check that

$$\ddot{R} = \frac{\Lambda}{3}, \tag{6.53}$$

which is easily seen.

6.4 Inhomogeneous factorizable solutions

In this section, I will deal with solutions where R factorize, but that are not homogeneous. The conditions from equation (6.40) are then

$$\alpha = k\omega \neq 0 \tag{6.54}$$

and

$$k \neq -\frac{1}{2}. \tag{6.55}$$

For now, cases where

$$\omega + \omega^2 = \frac{k}{1+2k} \tag{6.56}$$

will be tolerated as long as they obey the factorization identity

$$R(r,t) = a(t)R_i(r). \tag{6.57}$$

The Einstein equation (6.6) can be rewritten as

$$\dot{R}^2 + F = \frac{2k+1}{3}q(t)R^{2-\frac{6k}{2k+1}} + \frac{\Lambda}{3}R^2, \tag{6.58}$$

when using equation (6.15) and definition (6.22).

Utilizing factorization, this can be written as

$$\dot{a}^2 R_i^2 + F = \frac{2k+1}{3}q(t)(aR_i)^{2-\frac{6k}{2k+1}} + \frac{\Lambda}{3}(aR_i)^2. \tag{6.59}$$

Evaluating at the initial point in time gives

$$F = (-\dot{a}_i^2 + \frac{\Lambda}{3})R_i^2 + \frac{2k+1}{3}q_i R_i^{2-\frac{6k}{2k+1}} \equiv k_1 R_i^2 + k_2 R_i^{2-\frac{6k}{2k+1}}. \tag{6.60}$$

The distinction from the homogeneous case as well as the reduction to this when $\alpha = k = 0$ is readily seen. The curvature scalar, k_0, splits into two scalars, k_1 and k_2, where the latter represents curvature that is not constant in r.

As R_i counts as an initial condition, the radial dependence is more or less solved now, though a value for the initial Hubble parameter, $\dot{a}$ is also needed. One might be tempted to carry on to the analysis of the time dependence straight away, but the radial dependence still has something more to it that deserves attention. That is, from the radial dependence, the *shape* of the spatial sections can be deduced.

6.4.1 Spatial geometry

In the case of a FRW universe, the shape of spatial sections are easily seen from the sign of the curvature scalar. $k_0 < 0$ implies that the universe is open (mathematically speaking, not compact) with hyperbolic geometry on the spatial sections. $k_0 = 0$ gives flat, euclidean space (still not compact). Finally, $k_0 > 0$ is what characterizes closed (and bounded, hence compact) spatial sections. In this case, the spatial sections are spheres.

This becomes somewhat more subtle for inhomogeneous universes, as the curvature is not constant. One may begin this investigation by defining a position dependent curvature scalar as

$$k_0(r) = k_1 + k_2 R_i^{\frac{-6k}{2k+1}}, \tag{6.61}$$

such that

$$F = k_0(r) R_i^2 \tag{6.62}$$

and hence k_0 can be interpreted as the local value of the curvature. It is apparent that the curvature may take both negative and positive values if $k_1 \cdot k_2 < 0$.

It is worthwhile to repeat that the R_i of consideration obeys $R_i(0) = 0$ and $R_i'(r) > 0$ either for all r, or for all $r < r_0$ in which case $R_i'(r_0) = 0$.

This might potentially cause some trouble for the curvature scalar, as it diverges at $r = 0$ for $k < \frac{-1}{2}$ or $0 < k$ if $k_2 \neq 0$. However, I can do with a diverging curvature as long as the metric stays non singular. This less strict demand can be expressed as

$$k \in (-\frac{1}{2}, 1] \tag{6.63}$$

as $2 - \frac{6k}{2k+1}$ is required to take non-negative values. Note the physical meaning of this constraint. T^{rr} and $T^{\theta\theta} = T^{\phi\phi}$ are not allowed to have opposite signs. This restriction will be assumed in the following. Also, the preferred expression for F will be (6.60) rather than (6.62).

One should also recollect that for the metric to make sense,

$$R'(r_0) = 0 \Leftrightarrow F(r_0) = 1. \tag{6.64}$$

This means that the shape of the universe will depend strongly on whether there exists some r_0 for which $F(r_0) = 1$.

Moreover, from l'Hôpitals rule, it is found that

$$\lim_{r\to r_0} \frac{R_i'^2(r)}{1 - F(r)} = \lim_{r\to r_0} -\frac{2R_i'R_i''}{F'} = \lim_{r\to r_0} -\frac{2R_i''}{\frac{dF}{dR_i}} \tag{6.65}$$

This should be finite, hence

$$\frac{\partial F}{\partial R_i}(r_0) = 0 \Leftrightarrow R_i''(r_0) = 0. \tag{6.66}$$

The point of this is to check whether R_i' can be positive on both sides of r_0. If one assumes that R_i'' is continuous, then this can occur only if $R''(r_0) = 0$, as it is surely not positive.

Differentiating equation (6.60) with respect to R_i gives the condition for this peculiar case,

$$2k_1 R_i(r_0) + (2 - \frac{6k}{2k+1})k_2 R_i^{1-\frac{6k}{2k+1}}(r_0) = 0 \tag{6.67}$$

or

$$R_i(r_0) = (\frac{k_1}{k_2}\frac{2k+1}{k-1})^{-\frac{2k+1}{6k}}. \tag{6.68}$$

The consequences when this is the case will be clarified later on. For now, I will be content with noticing that if (6.68) is not true, then $R_i'(r) < 0$ for all $r > r_0$. This is seen as $R_i''(r_0) < 0$ and as F can be written as a function of R_i it will have no more possibilities to be equal to 1 at a later stage.

Leaving this peculiarity for a while, I return to the question of which cases permit $F = 1$. It is clear that solving the general equation

$$F(r_0) = k_1 R_i^2(r_0) + k_2 R_i^{2-\frac{6k}{2k+1}}(r_0) = 1 \tag{6.69}$$

for $R_i(r_0)$ in general is not feasible. However, one may easily find the conditions for when a positive solution exists. The different scenarios are illustrated in Figure 6.2

It is clear that if neither k_1 nor k_2 are positive, then no such solution exists. The curvature will be non-positive everywhere, giving a universe that is certainly not closed. This corresponds to the green line of figure 6.2

If both k_1 and k_2 are non-negative, there will surely be a solution except in one special case. That is, if $k = 1$ and $k_1 = 0$. If also $k_2 = 0$, this is just flat space. If $k_2 \neq 0$, then $F(0) \neq 0$. I demand that $F(r) = 0$ for a good reason. Sufficiently close to any point in space, the ratio between the circumference and radius of a circle centered at the given point should be 2π. Otherwise, the spatial manifold would not be differentiable. This is illustrated in figure 6.1.

The red line of figure 6.2 illustrates the case when either $k_i > 0$.

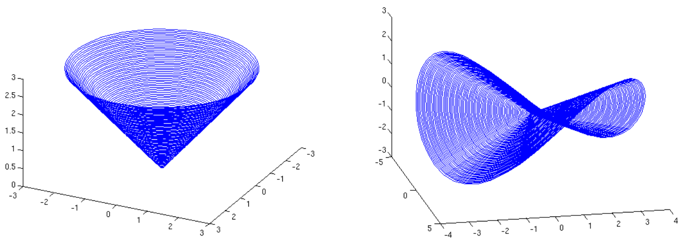


Figure 6.1: **Cone shaped spacetime** Two-dimensional spatial hyper-surfaces of spacetime corresponding to $k = 1, k_1 = 0, k_2 = \pm\frac{1}{2}$ are plotted. The origin is the tip/breakpoint of the cone and differentiability breaks down here.

The remaining cases are when one of the k_i is negative and the other one is positive. In these cases, F will always have an extremal value ($k \neq 0$ is assumed), when regarded as a function of R_i. I have already calculated the value of R_i when this extremal value occurs in equation (6.68). Define R_M as

$$R_M \equiv (\frac{k_1}{k_2}\frac{2k+1}{k-1})^{-\frac{2k+1}{6k}}. \tag{6.70}$$

This is seen to be real and positive in the interval $-\frac{1}{2} \leq k < 1$. Inserting this into equation (6.60) gives

$$F_M \equiv k_1(\frac{k_1}{k_2}\frac{2k+1}{k-1})^{-\frac{2k+1}{3k}} + k_2(\frac{k_1}{k_2}\frac{2k+1}{k-1})^{\frac{k-1}{3k}} \tag{6.71}$$

F_M is a maximum if $\underline{k_1 < 0 < k_2 \text{ and } -\frac{1}{2} < k < 0}$ or if $\underline{k_2 < 0 < k_1 \text{ and } 0 < k < 1}$. It is a minimum if either $\underline{k_1 < 0 < k_2 \text{ and } 0 < k < 1}$ or $\underline{k_2 < 0 < k_1 \text{ and } -\frac{1}{2} < k < 0}$.

In the latter case, F will certainly reach a value of 1 for some finite value of R_i. This corresponds to the cyan line of figure (6.2).

In the first case, F will take the value of 1 iff $F_M \geq 1$. One may recognise the limiting case of this, $F_M = 1$ as the case when equation (6.68) holds. Taking this one step further, it is obtained that this is true when

$$F_M = k_1\left(\frac{k_1}{k_2}\frac{2k+1}{k-1}\right)^{-\frac{2k+1}{3k}} + k_2\left(\frac{k_1}{k_2}\frac{2k+1}{k-1}\right)^{\frac{k-1}{3k}} = 1 \tag{6.72}$$

which can be solved for a positive k_1 as

$$k_1 = k_2\frac{k-1}{2k+1}\left(\frac{1}{k_2}\frac{k+1}{3k}\right)^{\frac{3k}{k-1}} \equiv k_{1c}, \tag{6.73}$$

or if k_2 is positive, for k_2 as

$$k_2 = k_1\frac{2k+1}{k-1}\left(\frac{1}{k_1}\frac{k-1}{3k}\right)^{\frac{3k}{2k+1}} \equiv k_{2c}. \tag{6.74}$$

It is clear that if F_M is a maximum for F and if k_i is positive, then there exists r_0 such that $F(r_0) = 1$ iff $k_i \geq k_{ic}$.

The three blue lines of figure 6.2 correspond to a case where there exists such an r_0, one case where there does not exist such an r_0, as well as the limiting case.

Now, the circumstances of how F depends on R_i has been somewhat clarified. Figure (6.2) provides an orderly overview. It is natural to follow up with how this relates to how R_i depends on r and the topology of the universe.

As has already been mentioned, R_i will be monotone on intervals where $F \neq 1$ and it will have a turning point if $F = 1$, except possibly if $\frac{\partial F}{\partial R_i} = 0$.
This gives an idea of how R_i will evolve in the different cases displayed in Figure 6.2. One may be inclined to believe that the curves which intersect the line $F = 1$ corresponds to universes where R_i is initially increasing, then eventually decreasing, reaching zero and closing the space. One is lead to think (and should rightfully do so) that the universes with $F < 1$ everywhere must have $R_i' > 0$ everywhere and thus the universe is unbounded. The case of the middle blue line where F approaches 1 as $\frac{\partial F}{\partial R_i}$ approaches zero is somewhat more subtle, but will be dealt with soon.

There is however one thing that ought to be checked in order to verify the above assertions. Although R_i' is strictly monotone, it may possibly converge to some limit as $r \to \infty$.

The way to get around this ambiguity is to demand that the spatial hypersurface has no boundary. I will demand that the spatial distance from $r = 0$ to some point at $r = r_1$ goes to infinity if r_1 goes to infinity.
The distance from the origin to a point with radial coordinate r is

$$\int_0^r \frac{R_i'(x)}{\sqrt{1 - F(x)}} dx. \tag{6.75}$$

If R_i is injective on $(0, r)$ this can be written

$$\int_0^{R_i(r)} \frac{1}{\sqrt{1 - F}} dR_i. \tag{6.76}$$

It is of course finite if $F \neq 1$ for every $R_i \in [0, R_i(r)]$. Whether it is finite or not is not so obvious if $F(r) = 1$. This case is taken care of by Theorem 1, equation (A.1)

The theorem is applicable to any situation where F is available as a function of R_i. If $F < 1$ everywhere, the distance from the origin to a point with radial coordinate r is finite as long as $R_i(r)$ is finite. As the universe is supposed to have no boundary, it must be infinite.
If $F(r_0) = 1$ and $\frac{\partial F}{\partial R_i}(r_0) \neq 0$, the universe will indeed be closed and if $\frac{\partial F}{\partial R_i}(r_0) = 0$, R_i will approach $R_i(r_0)$ infinitely far from the origin. The different spatial shapes have been plotted in figures 6.3(a)-(f). It must be stressed that they are not completely correct. The negatively curved parts get scaled up a bit. See appendix B for how they are produced.

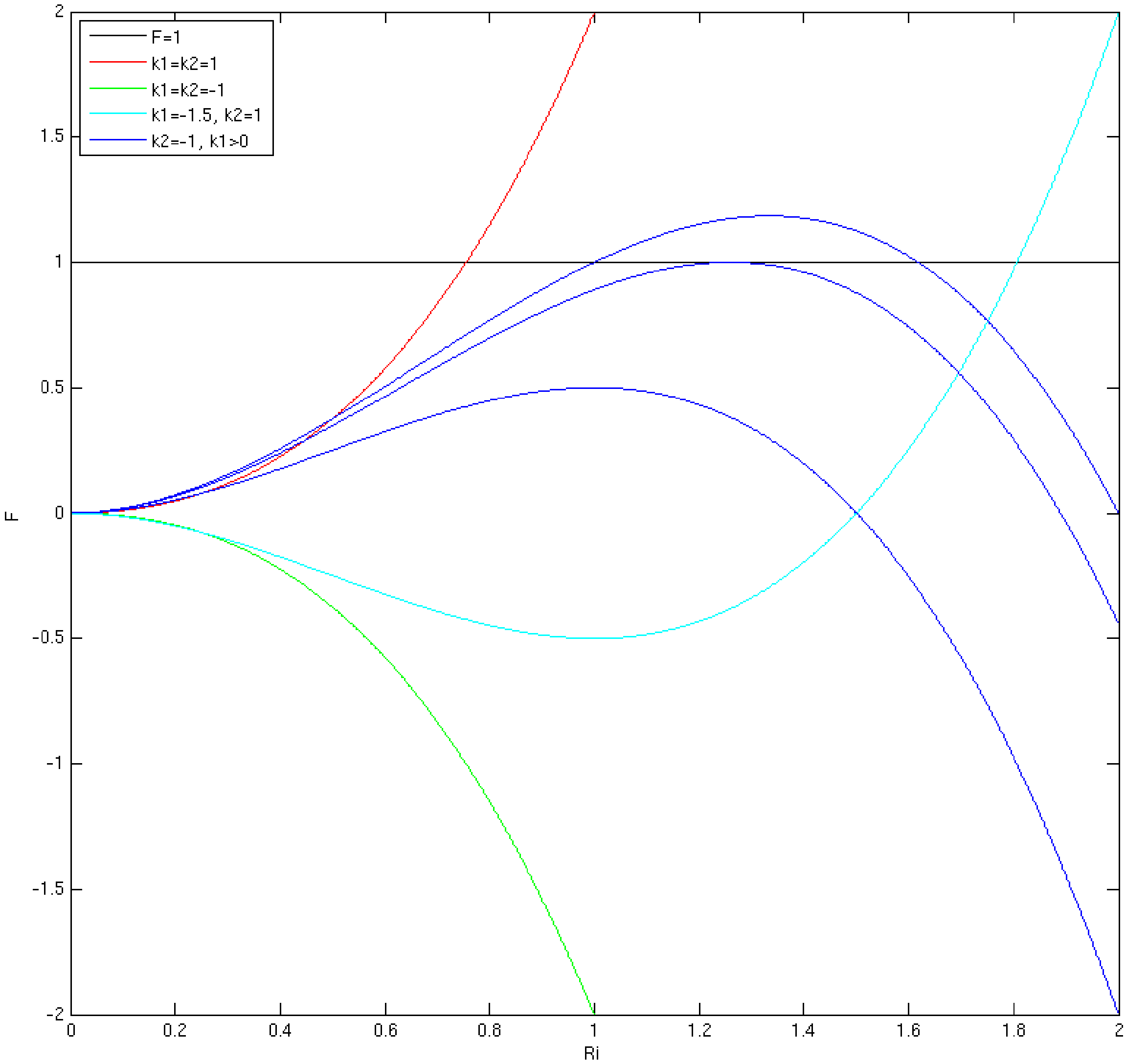
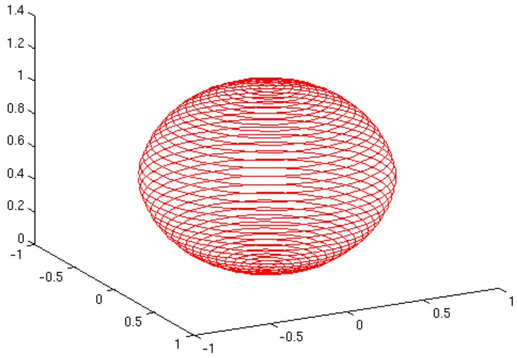
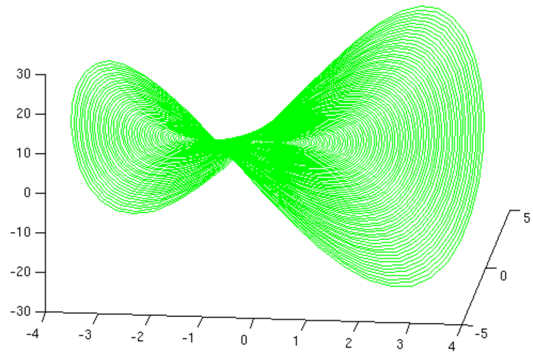


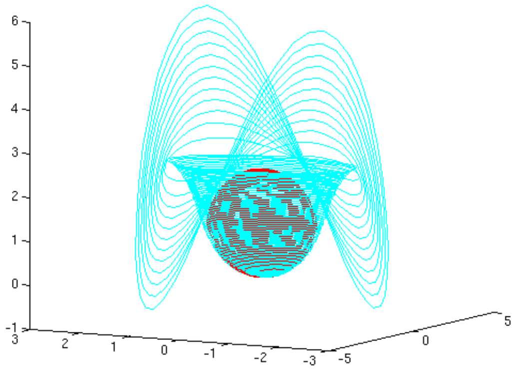
Figure 6.2: **F-R_i dependence** For every curve, $k = -\frac{1}{8}$, giving $F = k_1 R^2 + k_2 R^3$. The upper blue line has $k_1 = 2$, the limiting case has $k_1 = k_{1c} = \frac{3}{2} 2^{\frac{1}{3}}$, while the lowest blue has $k = \frac{3}{2}$.



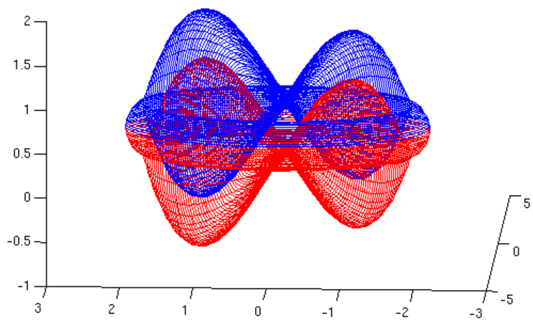
(a) $k_1 = k_2 = 1$



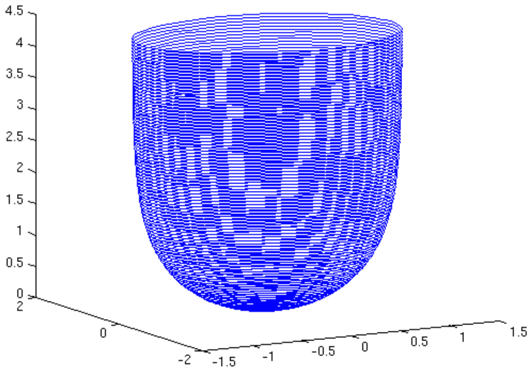
(b) $k_1 = k_2 = -1$



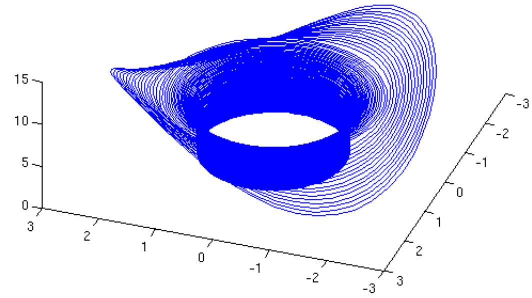
(c) $k_2 = -1,\ k_1 = k_c \pm 0.05$



(d) $k_1 = -1, 5,\ k_2 = 1$ (Cyan curve of figure 6.2)



(e) $k_2 = -1,\ k_1 = k_c$, first part



(f) $k_2 = -1,\ k_1 = k_c$, second part

Figure 6.3: Spatial sections, $k = -\frac{1}{8}$

The different shapes of figure 6.3 will be explained in further detail.
From equation (6.61) and the choice of $k = -\frac{1}{8}$ in the figures,

$$k_0 = k_1 + k_2 R_i. \tag{6.77}$$

The curvature depends linearly on R_i.

Figure 6.3(a) is topologically a sphere, but the curvature increases with R_i, making it slightly oblate.

The green surface of Figure 6.3(b) represent the exact opposite. It is negatively curved and the curvature grows more negative as R_i increases.

Figure 6.3(c) shows two different cases, corresponding to the blue curves of figure 6.2. The plotted shapes have been perturbed away from the middle line by a small amount. The spherical shape reaches $F = 1$ and hence it becomes closed. It is prolate due to the shrinking curvature with R_i. The upside down helmet that contains the sphere also has a curvature which shrinks with time. It follows the sphere closely for some time but it never closes, causing it to eventually become negatively curved.

Figure 6.3(d) shows the case of the cyan curve of figure 6.2. It is plotted in two colors in order to distinguish its two halves. This case has negative curvature in the origin, giving a saddle shaped area, the curvature eventually becomes positive, giving the "belt", the section eventually starts to grow inwards, taking the same path back on the way to closure.

The two last figures corresponds to two different universes given by the middle blue curve of figure 6.2. Figure 6.3(e) shows the part where R_i is less than the value giving $F = 1$. R_i converges to this value in the infinite. Figure 6.3(f) shows the other case, In this case there is no origin. R_i approaches the same value as in figure 6.3(e) but from above.

6.4.2 Time dependence

The other factor of $R = a(t)R_i(r)$, a(t) will be investigated in this subsection. Considering the consequences of factorization that were encountered in the chapters 4 and 5, it is not surprising that also here, factorization determines a lot. In particular, it will be shown that ω is very restricted.

Recall the form of the Einstein equation (6.59).

$$\dot{a}^2R_i^2 + F = \frac{2k+1}{3}q(t)(aR_i)^{\frac{-2k+2}{2k+1}} + \frac{\Lambda}{3}(aR_i)^2. \tag{6.78}$$

It is solved for F as

$$F = (\frac{\Lambda}{3}a^2 - \dot{a}^2)R_i^2 + (\frac{2k+1}{3}q(t)a^{\frac{-2k+2}{2k+1}})R_i^{\frac{-2k+2}{2k+1}}. \tag{6.79}$$

The time independence of F gives the constrictions

$$\frac{\Lambda}{3}a^2 - \dot{a}^2 = k_1 \tag{6.80}$$

and

$$\frac{2k+1}{3}q(t)a^{\frac{-2k+2}{2k+1}} = k_2. \tag{6.81}$$

From (6.81), it is obtained that

$$q(t) = \frac{3k_2}{2k+1}a^{\frac{2k-2}{2k+1}} \tag{6.82}$$

and

$$\rho = \frac{3k_2}{2k+1}R_i^{\frac{-6k}{2k+1}}a^{-2}. \tag{6.83}$$

The dependence $\rho \propto a^{-2}$ is associated with $\omega = -\frac{1}{3}$ and sure enough, inserting $\alpha = k\omega$ into equation (6.20) gives

$$\frac{\dot{\rho}}{\rho} = -3(\omega+1)\frac{\dot{a}}{a} \tag{6.84}$$

further inserting (6.83) gives

$$-2\frac{\dot{a}}{a} = -3(\omega+1)\frac{\dot{a}}{a}. \tag{6.85}$$

This means that the inhomogeneous factorizable solutions must obey $\omega = \frac{-1}{3}$, except if $\dot{a} = 0$. This means that ω may evolve freely in a static universe. From (6.80) this also implies that $a = \sqrt{\frac{3k_1}{\Lambda}}$. The more interesting dynamic solutions obey $p = -\frac{\rho}{3}$. Such perfect fluids are called **K-matter** [5].

One particular property of K-matter is that it gives a scale factor which increases linearly in time, hence the term *coasting cosmologies* for universes dominated by K-matter.

Integrating equation (6.80) gives

$$a = \sqrt{a_i^2 - \frac{\Lambda}{3k_1}} sinh(\pm\frac{\Lambda}{3}(t - t_i)) + a_i cosh(\pm\sqrt{\frac{\Lambda}{3}}(t - t_i)) \tag{6.86}$$

for $\Lambda \neq 0$ and

$$a = a_i + \sqrt{-k_1}(t - t_i) \tag{6.87}$$

for $\Lambda = 0$.

Interestingly enough, the constant growth of a K-matter dominated universe ($\Lambda = 0$) is carried on to the inhomogeneous, factorizable case. The extraordinary thing to notice is that the constant coasting emerges differently in the two cases.

With the FRW metric, the first Friedmann equation says

$$3\frac{\dot{a}^2 + k_0}{a^2} = \rho. \tag{6.88}$$

As ρ goes as a^{-2}, this simply gives

$$\dot{a}^2 = const. \tag{6.89}$$

The name "K-matter" is due to ρ acting in the same way as k_0.

In the inhomogeneous case, the term containing ρ depends differently on r and is not even included in equation (6.80)! The constant coasting in the inhomogeneous case is in fact due to the constant coasting of an empty universe. It may seem that the time evolution fits the type of fluid by a strike of luck!

Now that the factorizable solutions have been found, the next natural thing to do is to consider non-factorizable solutions. For example, one may want to consider the general case when $\omega^2 + \omega = \frac{k}{1+2k}$. For K-matter, this occurs if $k = -\frac{2}{13}$, but as these solutions don't factorize, one may also consider other sources.

The universe models which include radiation, dust and K-matter starts out in a radiation dominated era, eventually becomes dust dominated and ends up as K-matter dominated. If this is the case for our universe, the derivation of this chapter should give an idea of how a possibly inhomogeneous universe will look like in the future. Also, it can be interesting to approach the mixture of dust, radiation and K-matter in a similar way as in chapter 5.

This will need to be part of a future study.

Chapter 7

Conclusions and outlook

The thesis started out as a quest for investigating radiation dominated LTB-models. The motivation was that if the universe is significantly inhomogeneous in the present dust dominated era, it should have been inhomogeneous in the radiation dominated era as well.

It was shown in chapter 4 that any LTB model with radiation and a cosmological constant as the sole energy content must be homogeneous. It is unclear whether this was previously known or not as I have not found any articles on the subject. This does of course not mean that the universe must have been homogeneous in the radiative era, as the assumptions that are made are way to simplifying.

For example adding dust to the mixture of radiation and cosmological constant allows for inhomogeneity and this was studied in chapter 5. This chapter followed closely a previous work concerning radiation and dust, but no cosmological constant. I derived a differential equation governing the time evolution of the metric similar to the previous work, but also including a cosmological constant. Further, it was assumed that the cosmological constant was assumed to be zero, but dissimilar to the previous work, the curvature was not assumed to vanish. With this simplification, an exact solution for the time evolution of the metric was found and different types of evolution were analysed and plotted.

The puzzling discovery that radiation dominated LTB models reduce to homogeneous models leads to the question of whether this is the case for other sources as well. Chapter 6 analysed the much more general case of a general perfect fluid, $p = \omega\rho$ and a cosmological constant. More so, ω was allowed to vary in time and an anisotropic pressure proportional to ρ was included.

It was shown that for zero anisotropic pressure, the model reduced to a homogeneous one except in the cases $\omega = 0$ and $\omega = -1$. As far as I know this is a new result.

Furthermore, the conditions for factorization of the metric in a time-dependent and a position dependent factor was found in the general case with anisotropic pressure.

Every factorizable and inhomogeneous solution with this kind of source, except for dust was found and analyzed. It turned out that the only type of fluid that permitted such a solution was K-matter, a perfect fluid with $\omega = -\frac{1}{3}$.

Outlook

The results presented above raises several questions which invites to further investigation.

Merely postulating $p = \omega\rho$ with $0 \neq \omega \neq -1$ in the LTB metric reduced it to the FRW metric. It would be interesting to know if this is a part of a more general result, putting constraints also on more general metrics than the LTB metric.

As it was feasible to solve the model with dust and radiation, it should be possible to derive more general results with mixed fluids. In particular, it could be interesting to include K-matter.

The different equations of state that was assumed in chapter 6 give rise to many solutions which are not factorizable. A thorough analysis of these solutions or of subclasses of solutions is probably feasible. Also, even though the energy source in chapter 6 had several degrees of freedom, it could have been chosen even more general. A full categorization of every factorizable solution of the LTB metric would have been very interesting indeed.

Appendix A

A theorem

Theorem 1. *If F is a differentiable function $F : [0, r] \to [0, 1]$, $F(0) = 0,\ F(r) = 1$, with $F'(x) > 0\ \forall\ x\ \in\ [0, 1)$, then*

$$\int_0^r \frac{dx}{\sqrt{1 - F(x)}} \tag{A.1}$$

converges if and only if $F'(r) \neq 0$

Proof. Assume $F'(r) \neq 0$ and hence $F'(x) > 0 \forall x$. This implies that F is injective and one can make a change of variables. Also let $F'_{min} = \inf_{x \in [0,r]}(F'(x))$. This gives

$$\int_0^r \frac{dx}{\sqrt{1 - F(x)}} = \int_0^1 \frac{-dF}{F'(x(F))} \frac{1}{\sqrt{1 - F}} \leq \frac{1}{F'_{min}} \int_0^1 \frac{-dF}{\sqrt{1 - F}} = 2\frac{1}{F'_{min}} < \infty \tag{A.2}$$

For the direction of the inequality, one must observe that the integral is strictly positive.
For the converse, assume $F'(r) = 0$. Using the fact that F is increasing gives the inequality

$$\frac{r}{n}\frac{1}{\sqrt{1 - F(r\frac{n-1}{n})}} \leq \frac{r}{n}\sum_{i=0}^{n-1} \frac{1}{\sqrt{1 - F(r\frac{i}{n})}} < \int_0^r \frac{dx}{\sqrt{1 - F(x)}}. \tag{A.3}$$

The assumption $F'(1) = 0$ can be expressed in a more formal manner as

$$\forall\ \epsilon > 0\ \exists\ \delta > 0 \mid \forall\ 0 < h < \delta,\ \frac{1}{h}(F(r) - F(r - h)) < \epsilon. \tag{A.4}$$

Let M be an arbitrary positive number. If $\epsilon = \frac{1}{M^2}$, then there exists $\delta > 0$ such that if $0 < h < \delta$, then $\frac{1}{h}(F(r) - F(r - h)) < \epsilon$. Choose some such h with the property that $h < 1,\ h < r$ and $h = \frac{r^2}{n^2}$ for some natural number n. Observing that $F(r-h) > F(r-\sqrt{h})$ gives the inequality

$$\frac{n^2}{r^2}(1 - F(r - \frac{r}{n})) < \frac{n^2}{r^2}(1 - F(r - \frac{r^2}{n^2})) < \epsilon. \tag{A.5}$$

This is suitably rewritten as

$$\frac{r}{n}\frac{1}{\sqrt{1 - F(r\frac{n-1}{n})}} > M \tag{A.6}$$

Comparing with equation (A.3) gives

$$M < \int_0^r \frac{dx}{\sqrt{1 - F(x)}}, \tag{A.7}$$

completing the proof.

$\square$

Appendix B

Illustrative embeddings

The spatial sections of inhomogeneous universe models come in a variety of different shapes. For purposes of perception and illustration, one may want to plot these shapes.

The spatial sections are in general three dimensional curved spaces. In order to see the curvature, the spatial sections must be embedded in a space of a higher dimension.

Plotting in 4 dimensions is of course not feasible, but one may use that the space is spherically symmetric in order to simplify. If for example the θ coordinate is fixed, the two dimensional surface that remains will still say a lot about the shape of the space. This is what I will do. The surfaces that will be plotted are the

$$t = 0,\ \theta = \frac{\pi}{2} \tag{B.1}$$

subspaces of different models. These surfaces are plotted in a 3D plot using matlab.

B.1 Embedding the surface

In this section it is explained how I assign to the surface its coordinates in $\mathbb{R}^3$ which are *locally* consistent with the curvature. Embedding spaces of partially positive and partially negative curvature leads to some problems and it will be clear in the following that I deliberately make an inaccuracy.

It is necessary to distinguish between $F > 0$ and $F < 0$. These cases will give different rules for embedding. The case when $F = 0$ will be the same in both sets of rules and this makes it possible to glue a positively curved part seamlessly together with a negatively curved part.

I choose to place the origin of the universe surface in the origin of $\mathbb{R}^3$. Also, the x, y coordinates of the surface $x = R_i cos(\phi), y = R_i sin(\phi)$ will coincide with the x, y coordinates of the embedding space. The z coordinate will vary according to the curvature.

For non-negative curvature, $F \geq 0$ one may preserve the symmetry in ϕ and draw the surface as a solid of revolution. The question becomes how the z coordinate varies with r. The answer to this is found when observing that the traversed distance in $\mathbb{R}^3$ when moving in the radial direction must equal the distance traversed in the universe model. That is

$$(z'^2 + R_i'^2)dr^2 = \frac{R_i'^2}{1-F}dr^2 \tag{B.2}$$

This can be solved for z', however, I will use a different approach.
In the computer program, I plot the surface as a series of equidistant lines. The distance will be the left hand side of equation (B.2) and will be assumed to be small. Denoting this distance by h and writing differentials as finite sizes gives

$$\Delta R_i = h\sqrt{1-F} \tag{B.3}$$

and

$$\Delta z = h\sqrt{F}. \tag{B.4}$$

This gives the difference equations

$$(R_i)_{n+1} = (R_i)_n + h\sqrt{1 - F((R_i)_n)} \tag{B.5}$$

and

$$(z)_{n+1} = (z)_n + h\sqrt{F(R_i)_n}, \tag{B.6}$$

where F is understood as a function of R_i. The initial conditions are

$$(R_i)_1 = (z)_1 = 0. \tag{B.7}$$

For non-positive curvature, it is somewhat more complicated, and the method I have chosen here is not exact. The negatively curved areas will be scaled differently from the areas of non negative curvature. One should have this in mind when studying the plots, but the local shape is at least consistent with the curvature, and this justifies the use of the illustrations.

It is clear that the difference equation (B.6) becomes complex for negative F.

As a first approximation, let $\{R_i\}_n$ and $\{z\}_n$ behave as if $F = 0$. This gives the difference equations

$$(R_i)_{n+1} = (R_i)_n + h \tag{B.8}$$

and

$$(z)_{n+1} = (z)_n. \tag{B.9}$$

This is just the difference equation for flat space.
If the metric is multiplied with $1 - F$, the difference equation for R_i adapts this form. However, when doing so, the circumference at R_i is now $2\pi\sqrt{1-F}R_i$. I deal with this by adding a ϕ dependent term to z.

Inspired by the curvature of the surface $z = xy$, this ϕ dependent term will be on the form

$$(\Delta z)_n = (axy)_n = a_n(R_i)_n cos(\phi)sin(\phi). \tag{B.10}$$

When plotting the surfaces, $(z)_n + (\Delta z)_n$ will be the z coordinate.

The parameter a will be decided from the condition that the circumference is $\sqrt{1-F}2\pi R_i$. Observe from (B.10) that a has absorbed a factor of R_i when compared to $z = xy$.

$$\int_0^{2\pi} \sqrt{1 + (\frac{d\Delta z}{d\phi})^2} d\phi = \int_0^{2\pi} \sqrt{1 + a^2 cos^2(2\phi)} d\phi = 2\pi\sqrt{1-F} \tag{B.11}$$

This integral will be solved numerically in the subsequent section.

B.2 Auxiliary programs

In order to implement the algorithm presented above, one needs some way to calculate the parameter a. Also, implementing functions will make the program more orderly.

B.2.1 $F(R_i)$

The following function simply calculates $F(R_i)$

```
function ret=F(R,k1,k2,k)
ek=2-6*k/(2*k+1);
ret=k1*R.^2+k2*R.^ek;
end
```

B.2.2 Numerical integration

The integral (B.11) is calculated numerically by dividing the range of ϕ into n equally spaced sub-intervals and making the trapezoid approximation.

```
function A=inte(a)
n=1000;
A=0;
x=linspace(0,2*pi,n+1);
for i=1:n
    A=A+2*pi/n*sqrt(1+a*a*cos(2*x(i))^2);
end

end
```

B.2.3 Binary search

The integral in equation (B.11) is seen to be strictly increasing with the absolute value of a. This can be utilized to make a binary search for the value of a which satisfies the equation. This is done as follows.

```
function A=kon(F)
A=0;
if (F<0)
    q=2*pi*sqrt(1-F);
    a=0;
    b=1;
    while (inte(b)<q)
        a=b;
        b=2*b;
    end
    au=inte(a);
    bu=inte(b);
    c=(a+b)/2;

    while (c-a>0.001)
        y=inte(c);
        if (y<q)
            a=c;
```

```
            au=y;
        end
        if (y>=q)
            b=c;
            bu=y;
        end
        c=(a+b)/2;
    end
    A=c;
end
```

B.3 The program

The following matlab code implements the difference equations (B.5-B.10). It applies the auxiliary functions of the preceding section.

```
% variables
k=-1/8;
k1=1;%1.5*2^(1/3);
k2=1;
closed=1; % 1 if the surface is closed, 0 if it is open
h=.05; %distance between neighbouring curves
m=50; %Number of ponts in angular direction
Nopen=150; %The number of steps if closed=0

%initialization
N=0;
R(1)=0;
z(1)=0;
if (closed==1) % in case of a closed surface
    while(F(N*h,k1,k2,k)<1)
        N=N+1;
    end
    RM=h*(N-1); %Maximal value of Ri
    N=1;
    while(R(N)<RM)
        f=F(R(N),k1,k2,k);
        if (f<=0)
            R(N+1)=R(N)+h;
            z(N+1)=z(N);
        end
        if (f>0)
            R(N+1)=R(N)+h*sqrt(1-f);
            z(N+1)=z(N)+h*sqrt(f);
        end
        N=N+1;
```

```
    end
    for i=1:N %making a mirrored copy
        R(N+i)=R(N-i+1);
        z(N+i)=h+2*z(N)-z(N-i+1);
    end
    N=2*N;

end
if (closed==0)
    N=Nopen;
    for i=1:N-1
        f=F(R(i),k1,k2,k);
        if (f<=0)
            R(i+1)=R(i)+h;
            z(i+1)=z(i);
        end
        if (f>0)
            R(i+1)=R(i)+h*sqrt(1-f);
            z(i+1)=z(i)+h*sqrt(f);
        end
    end
end
hold off

x=cos(linspace(0,2*pi,m));
y=sin(linspace(0,2*pi,m));

for i=1:N %calculating Delta z and plotting
    f=F(R(i),k1,k2,k);
    if (f<0)
        a=kon(f);
    end
    if (f>=0)
        a=0;
    end
    plot3(R(i)*x,R(i)*y,z(i)+a*R(i)*x.*y,'r')
    hold on
end
```

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